Solutions to K. Becker, M. Becker, J. Schwarz *String Theory And M-theory*

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Abstract

Detailed solutions to 154 of 182 of homework problems in K. Becker, M. Becker, J. Schwarz *String Theory And M-theory* textbook are presented.

1 Introduction

Studying the course of String Theory and solving these problems I have extensively used textbooks of BBS [1]; GSW [2]; Polchinski [3]; Kaku [4]; Di Francesco, *et al.* [5]; Kiritsis [6]; Hawking, Ellis [7]; S. Weinberg [8], etc. Some of the problems given in BBS as homework task are actually well known string theory facts which are described in papers or textbooks, especially in GSW.

References to equations of some textbook are given here in format book (formula), e.g. BBS (12.140) gives metric for an extremal black $D3$-brane. Equations without any author acronym in front of it refer to the present paper.

This work was done by myself while I was fifth year undergraduate student at MIPT on my Master Program and was working in BLTP JINR and carries no endorsement from K. Becker, M. Becker or J.H. Schwarz.

2 The bosonic string

`Problem 2.1`

(i) String equations of motion in conformal gauge of world-sheet metric (flat metric for world-sheet with no topological obstructions) are

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu = 0. \tag{2.1}$$

These equations are satisfied by the following open string classical configuration

$$X^0 = B\tau, \quad X^1 = B\cos\tau\cos\sigma,$$

$$X^2 = B\sin\tau\cos\sigma, \quad X^i = 0,\ i > 2.$$

Obviously Neumann boundary conditions

$$X'^\mu(\sigma = 0, \pi) = 0 \tag{2.2}$$

are satisfied too.

3-velocity of some point on string is equal to

$$v^i = \frac{dX^i}{dX^0} = \frac{1}{B}\frac{dX^i}{d\tau},$$

modulus of which on the ends of the treated string is evidently equal to 1, and therefore the ends of this string are indeed moving with the speed of light.
(ii) As an analogy to the point particle we can write D-momentum of points of string (we can use Noether theorem too, and build energy-momentum tensor, from which we can get this σ-density of 4-momentum):

$$P^\mu_\alpha = T\partial_\alpha X^\mu. \tag{2.3}$$

From this expression we can find the total energy of string

$$E = \int d\sigma P^0_0 = \pi BT.$$

We can use Noether theorem to derive the density of a conserved angular momentum tensor

$$J^{\mu\nu} = T\left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right) \tag{2.4}$$

and use it to derive total angular momentum of the considered string:

$$J = \int d\sigma |J_3| = \int d\sigma J^{12} = \frac{1}{2}TB^2.$$

Obviously it takes place an equality

$$\frac{E^2}{J} = 2\pi T.$$

(iii) In conformal gauge the constraint $T_{\alpha\beta} = 0$, depicting equations of motion for world-sheet metric, may be rewritten as

$$\dot{X}^2 + (X')^2 = 0, \quad \dot{X} \cdot X' = 0,$$

which is evidently satisfied by the considered open string configuration.

Problem 2.2
(i) It's very easy to show that the open string configuration

$$X^0 = 3A\tau, \quad X^1 = A\cos(3\tau)\cos(3\sigma),$$

$$X^2 = A\sin(a\tau)\cos(b\sigma), \quad X^i = 0,\ i > 2.$$

satisfies energy-momentum constraint $T_{\alpha\beta} = 0$ if

$$a = b = \pm 3.$$

Obviously here Neumann boundary conditions are satisfied.

Note, that there's another fact that approves that $b = \pm 3$. This is due to the known general solution for an open string with Neumann boundary conditions

$$X^\mu(\tau, \sigma) = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m\neq 0} \frac{1}{m}\alpha^\mu_m e^{-im\tau}\cos(m\sigma). \tag{2.5}$$

This expression may be used directly to show that center-of-mass momentum p^μ and center-of-mass coordinate x^μ are equal to zero for considered string configuration. But assuming no general formulae are known this result will be discussed in (iii).

Let's take a look at how considered string configuration is written in light-cone coordinates

$$\sigma^\pm = \tau \pm \sigma. \tag{2.6}$$

It's easy to calculate that this string space-time configuration divides in two parts - left-movers and right-movers:

$$X^\mu(\sigma^+, \sigma^-) = X^\mu_L(\sigma^-) + X^\mu_R(\sigma^+),$$

with the following expression being hold:

$$X^\mu_L(\sigma^-) = \left(\frac{3A}{2}\sigma^-,\ \frac{A}{2}\cos(3\sigma^-),\ \frac{A}{2}\sin(a\sigma^-)\right),$$

$$X^\mu_R(\sigma^+) = \left(\frac{3A}{2}\sigma^+,\ \frac{A}{2}\cos(3\sigma^+),\ \frac{A}{2}\sin(a\sigma^+)\right).$$

Neumann boundary conditions here are represented as

$$\frac{\partial X^\mu_L}{\partial\sigma^-} + \frac{\partial X^\mu_R}{\partial\sigma^+} = 0,\ \sigma = 0, \pi.$$

(ii) Plot will not be drown here, but a brief description is pretty easy. The string is a polyline, which is a 3-diameter length line, the motion of which is rotation over the circle with mentioned diameter.

(iii) Center-of-mass momentum together with angular momentum may be directly computed with the help of their expressions due to Noether theorem:

$$P^\mu = T\int d\sigma \dot{X}^\mu = (3\pi AT, 0, \ldots, 0),$$

$$J = \int d\sigma |J_3| = \int d\sigma |J^{12}| = \frac{3\pi}{2}TA^2.$$

Now energy and absolute value of momentum are related to each other as $\frac{E^2}{|J|} = 6\pi T$. Well, at least now we have higher excitation of string than in previous problem.

`Problem` **2.3**

(i) For fields $X^\mu,\ \mu = 0, \ldots, 24$ open string with Neumann boundary condition is described by expression (2.5). Because of equations of motion for string moving in flat space-time are independent for various space-time coordinate fields on world-sheet, we should just separately treat how Dirichlet boundary conditions

$$X^{25}(0,\tau) = X^{25}_0, \quad X^{25}(\pi,\tau) = X^{25}_\pi \tag{2.7}$$

influence the solution of string wave equation of motion. In the theory of Equations of Mathematical Physics the method of searching a Fourier expansion of wave equation with some chosen boundary conditions is developed, which may be exploited here to write down the solution:

$$X^{25} = \frac{\sigma}{\pi}\left(X^{25}_\pi - X^{25}_0\right) + X^{25}_0 + il_s\sum_{m\neq 0}\frac{1}{m}\alpha^{25}_m e^{-im\tau}\sin(m\sigma).$$

Momentum conjugate to this coordinate may be calculated by formula (2.3), but this is no more a Noether expression (just an ordinary rule for computation of momentum conjugate to coordinate) because Dirichlet condition breaks X^{25} condition translation symmetry. It may be easily found that

$$P^{25} = 2Tl_s \sum_m \frac{1}{2m+1} \alpha^{25}_{2m+1} e^{-i(2m+1)\tau}.$$

This momentum is not conserved: string is impressed by an effective force of reaction in X^{25} direction, which holds X^{25} coordinate fixed. Momentum is oscillating and force is oscillating too.

(ii) We can formulate the answer immediately:

$$X^{25}(\tau,\sigma) = X^{25}_0 + il_s \sum_m \frac{2}{2m+1} \alpha^{25}_m e^{-i\frac{2m+1}{2}\tau} \sin\left(\frac{2m+1}{2}\sigma\right),$$

while the rest 25 coordinates are governed by equation (2.5). Non-conserved conjugate momentum is equal to

$$P^{25} = Tl_s \sum_m \frac{2}{2m+1} \alpha^{25}_m e^{-i\frac{2m+1}{2}\tau}.$$

Problem 2.4

(i) We will use the open string mass formula

$$M^2 = \frac{2}{l_s^2}(N-1). \tag{2.8}$$

It's easy to check that

$$N|\phi_b\rangle = b|\phi_b\rangle,\ \ b = 1,2,3,4,$$

therefore

$$M_b^2 = \frac{2}{l_s^2}(b-1).$$

(ii) We will use the closed string mass formula

$$M^2 = \frac{8}{l_s^2}(N-1) = \frac{8}{l_s^2}(\tilde{N}-1). \tag{2.9}$$

Here we got

$$N|\phi_b\rangle = \tilde{N}|\phi_b\rangle = b|\phi_b\rangle,\ \ b = 1,2,$$

therefore

$$M_b^2 = \frac{8}{l_s^2}(b-1).$$

(iii) Such state violates level-matching constraint of the bosonic string, and therefore it contradicts to vanishing of world-sheet energy momentum tensor.

Problem 2.5

We employ commutation relations, postulated in quantum theory

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0}, \quad [\alpha^\mu_m, \tilde{\alpha}^\nu_n] = 0, \tag{2.10}$$

which are actually defined for $m, n \neq 0$, but can be literally generalized for possible zero indices, where $\alpha_0^\mu = \frac{1}{2} l_s p^\mu$. Then it's used a formula

$$\delta(\sigma - \sigma') = \frac{1}{\pi} \sum_m \cos(m\sigma) \cos(m\sigma') \tag{2.11}$$

which is delta-function representation in terms of real Fourier series. Indeed,

$$\int d\sigma' \delta(\sigma - \sigma') f(\sigma') = \frac{1}{\pi} \sum_m \cos(m\sigma) \int d\sigma' f(\sigma') \cos(m\sigma') = \sum_m f_m \cos(m\sigma) = f(\sigma). \tag{2.12}$$

the limits of integration here depend on whether function is periodic (then the period of function is a realm of integration) or defined on finite region of σ'-axis (then it is integrated over the whole real axis). In both cases there're mathematical propositions helping us to perform back and forward Fourier transformations by such kind of a formulae. By the way (it doesn't have relation to this problem), complex Fourier expansion may be used to prove in the same way corresponding delta-function representation

$$\delta(\sigma) = \frac{1}{\pi} \sum_m e^{2im\sigma}. \tag{2.13}$$

Density of momentum is a D-momentum field on world-sheet:

$$P^\mu = T\dot{X}^\mu = T \left(l_s^2 p^\mu + l_s \sum_{m \neq 0} \alpha_m^\mu e^{-im\tau} \cos(m\sigma) \right) = l_s \sum_m \alpha_m^\mu e^{-im\tau} \cos(m\sigma).$$

It can be easily calculated that

$$[X^\mu(\sigma, \tau), X^\nu(\sigma', \tau)] = l_s^2 \eta^{\mu\nu} \sum_{m \neq 0} \frac{1}{m} \cos(m\sigma) \cos(m\sigma') = 0.$$

The last transition is made because in that sum we got all terms with their negatives. Similarly

$$[P^\mu(\sigma, \tau), P^\nu(\sigma', \tau)] = T^2 l_s^2 \eta^{\mu\nu} \sum_{m \neq 0} m \cos(m\sigma) \cos(m\sigma') = 0.$$

With the help of (2.11) it can be easily shown that

$$[X^\mu(\sigma, \tau), P^\nu(\sigma', \tau)] = i\eta^{\mu\nu} \delta(\sigma - \sigma'). \tag{2.14}$$

`Problem` **2.6**

First, by definition center-of-mass coordinates are α_m-independent (for $m \neq 0$), because α_m, $m \neq 0$ values depicts string oscillating terms, which after averaging procedure (giving center-of-mass values) return zero values. Commutator for momentum and coordinate functions on world-sheet (σ-densities) (2.14) after σ and σ' integration $\left(\int d\sigma X^\mu = \pi x^\mu\right)$, $\int d\sigma P^\mu = p^\mu$ gives

$$[p^\mu, x^\nu] = -i\eta^{\mu\nu}.$$

Using this relation we can easily prove the following:

$$[p^\mu, J^{\nu\sigma}] = -i\eta^{\mu\nu} p^\sigma + i\eta^{\mu\sigma} p^\nu.$$

To prove an expression for Lorentz generator commutator there's the simplest way: first - to prove it for center-of mass part $x^\mu p^\nu - x^\nu p^\mu$, second - to claim that Lorentz transformations as transformations of symmetry form a group, therefore oscillating terms in commutator should be gathered in an appropriate way (this is not rigorously mathematically but reasonably physically).

`Problem` **2.7**

We will exploit the following formula for delta-symbol representation:

$$\int_0^\pi d\sigma e^{2i\sigma n} = \pi \delta_{n,0}$$

which may be obviously verified by direct calculation. Anyway, for closed string general world-sheet configuration is

$$X^\mu = X_R^\mu + X_L^\mu = x^\mu + l_s^2 p^\mu \tau + \frac{il_s}{2} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \right). \tag{2.15}$$

From this expression we can easily obtain

$$\dot{X}^\mu = l_s^2 p^\mu + l_s \sum_{n\neq 0} \left(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \right).$$

After not too difficult and not too long calculations we will get

$$J^{\mu\nu} = T\int_0^\pi d\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu \right) = x^\mu p^\nu - x^\nu p^\mu + \frac{i}{2} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu \alpha_{-n}^\nu - \alpha_n^\nu \alpha_{-n}^\mu + \tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu - \tilde{\alpha}_n^\nu \tilde{\alpha}_{-n}^\mu \right).$$

Cross-terms with α and $\tilde{\alpha}$ have canceled each other after σ-integration, that is substitution of delta-symbol representation. The last formula may be rewritten in normal-ordered form (commutators cancel each other)

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^\infty \frac{1}{n} \left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu + \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu \right).$$

`Problem` **2.8**

In light-cone gauge we got $\alpha_n^+ = 0$, which lets us to write down Lorentz generators

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^\infty \frac{1}{n} \left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right)$$

in light-cone coordinates as follows:

$$J^{+-} = x^+ p^- - x^- p^+, \quad J^{\pm i} = x^\pm p^i - x^i p^\pm,$$

$$J^{ij} = x^i p^j - x^j p^i - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n\right)$$

`Problem` **2.9**

The easiest way to perform calculations in this problem is: first - to separate oscillating and center-of-mass parts, second - to consider general view of oscillating terms in commutator $[J^{\mu\nu}, L_m]$. Suppose we perform a summation over oscillating terms in L_m with index k and over oscillating terms of $J^{\mu\nu}$ with index n. General view of oscillating term in commutator is

$$-i\frac{1}{2n}[\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n, \alpha^\lambda_{m-k}\alpha^\sigma_k\eta_{\lambda\sigma}].$$

After performing summation over k we will result in

$$-i\left(\alpha^\nu_{m-n}\alpha^\mu_n - \alpha^\mu_{m-n}\alpha^\nu_n + \alpha^\mu_{-n}\alpha^\nu_{m+n} - \alpha^\nu_{-n}\alpha^\mu_{m+n}\right).$$

Then we should perform a summation over n, change some summation indices $n \to -n$, permute some multipliers in a couple of terms, commutators of which will cancel each other. As a result we will get

$$-i\left(\sum_{n>1}\alpha^\nu_{m-n}\alpha^\mu_n + \sum_{n<-1}\alpha^\nu_{m-n}\alpha^\mu_n\right) + i\left(\sum_{n>1}\alpha^\mu_{m-n}\alpha^\nu_n + \sum_{n<-1}\alpha^\mu_{m-n}\alpha^\nu_n\right).$$

Now we can add and subtract $n = 0$ terms and get the result

$$-i\left(\sum_{n=-\infty}^{\infty}\alpha^\nu_{m-n}\alpha^\mu_n - \alpha^\mu_{m-n}\alpha^\nu_n\right) - i(\alpha^\mu_m\alpha^\nu_0 - \alpha^\nu_m\alpha^\mu_0)$$

First sum is equal to zero. We can just change $n \to m - n$ summation in one of the terms to get it. The rest terms cancel with

$$\left[x^\mu p^\nu - x^\nu p^\mu,\ \frac{1}{2}\sum_n \alpha_{m-n}\cdot\alpha_n\right]$$

where non-zero commutators will be only for $n = 0$, m as $\sim [x^\mu, p^\lambda]$.

One concludes that every physical state is brought again to physical state by Lorentz transformation, moreover the mass of state doesn't change (the same is true about spin, which is Casimir for Lorentz group). Therefore physical states are grouped into Lorentz multiplets.

`Problem` **2.10**

Consider first generalized mass-shell condition on physical state

$$(L_0 - a)|\phi\rangle = \left(\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \frac{1}{2}\alpha_0^2 - a\right)|\phi\rangle = 0.$$

Here $a = 1$ and simple computation allows us to rewrite this condition as

$$\left(1 + \frac{1}{2}\alpha_0^2\right)|0; k\rangle = 0,$$

from which it follows that $\alpha_0^2 = -2$ - the equation on mass of the open string in the state $|\phi\rangle$.

Second, consider Virasoro constraints $L_m|\phi\rangle = 0,\ m > 0$. Use definition $L_m = \frac{1}{2}\sum_n \alpha_{m-n} \cdot \alpha_n$ and consider separately terms with $n > 0,\ n < 0,\ n = 0$. Act with such an operator on $|\phi\rangle$. For $m > 2$ constraints are satisfied automatically. Cases $m = 1$ and $m = 2$ give us

$$L_1|\phi\rangle = 2(A + B + C\alpha_0^2)\alpha_0 \cdot \alpha_{-1}|0;k\rangle, \quad L_2|\phi\rangle = (DA + 2B\alpha_0^2 + C\alpha_0^2)|0;k\rangle,$$

and corresponding physical state conditions

$$A + B + C\alpha_0^2 = 0, \quad DA + 2B\alpha_0^2 + C\alpha_0^2 = 0,$$

which are solved by

$$B = \frac{D-1}{5}A, \quad C = \frac{D+4}{10}A.$$

Different oscillating states are obviously orthogonal to each other. Therefore the norm of the state $|\phi\rangle$ is equal to

$$\langle\phi|\phi\rangle = 2(DA^2 - 2B^2 - 4AC + 4C^2) = \frac{2A^2}{25}(D-1)(26-D).$$

The norm is negative for $D > 26$ and zero for $D = 26$.

`Problem` **2.11**

Closed string states with excitation numbers $N = \tilde{N} = 2$ are built out of vacuum state $|0;k\rangle$ by acting with rising operators

$$\alpha_{-1}^i\alpha_{-1}^j,\ \alpha_{-2}^i,\ \tilde{\alpha}_{-1}^i\tilde{\alpha}_{-1}^j,\ \tilde{\alpha}_{-2}^i.$$

Such a state is massive, therefore each index is related to representation of Lorentz little subgroup $SO(25)$. Left-movers and right-movers parts form independently rank-two tensor representation of $SO(25)$ with dimension $\frac{24\cdot 25}{2} = 324$. The whole state forms representation of $SO(25) \times SO(25)$.

`Problem` **2.12**

For open string $N = 3$ states are

$$\begin{aligned} &\alpha_{-1}^i\alpha_{-1}^j\alpha_{-1}^k|0;k\rangle && \frac{24\cdot 23\cdot 22}{6} + 24\cdot 23 + 24 = 2600 \text{ states;} \\ &\alpha_{-2}^i\alpha_{-1}^j|0;k\rangle && 24^2 = 576 \text{ states;} \\ &\alpha_{-3}^i|0;k\rangle && 24 \text{ states.} \end{aligned}$$

Total number of states is 3200.

Let's explore decomposition of these states into irreducible representation of massive little group $SO(25)$. First of all note that as was discussed in the solution of Problem 2.9 physical states group into Lorentz multiplets. Then observe that symmetric traceless rank 3-tensor with $SO(25)$ indices has $2925 - 25 = 2900$ components. Here 2925 is the number of components of symmetric rank-3 tensor with each index of it taking 25 values, which may be derived either as shown above for 24 values of each index in the context of number of states, or as dimension of Young table with three horizontal squares. We take traceless tensors - this is expressed as $\Omega_{iik} = 0$ for all 25 values of k (we can't sum three equal indices, but nevertheless we may perform a little generalization: denote $\omega_k = \Omega_{iik}$, then due to imposed constraints one will have $\sum_k \omega_k = 0$) - this condition is Lorentz-invariant and therefore result in 2900 components. Until this moment we were acting in a way described in BBS for $N = 2$ level. We shall now add 300 more states, which is the number rank-2 antisymmetric tensor components. Therefore we result in decomposition $\mathbf{2900} + \mathbf{300}$.

For closed string $N = \tilde{N} = 3$ states each of described $N = 3$ states is accompanied with some of the $\tilde{N} = 3$ state out of

$$\tilde{\alpha}^i_{-1}\tilde{\alpha}^j_{-1}\tilde{\alpha}^k_{-1}|0;k\rangle \qquad \frac{24\cdot 23\cdot 22}{6} \ + \ 24\cdot 23 \ + 24 \ = \ 2600 \text{ states};$$

$$\tilde{\alpha}^i_{-2}\tilde{\alpha}^j_{-1}|0;k\rangle \qquad 24^2 \ = \ 576 \text{ states};$$

$$\tilde{\alpha}^i_{-3}|0;k\rangle \qquad 24 \text{ states}.$$

The structure of states is the same as for right-movers above with additional (anti)symmetrization between left- and right-movers where it's possible. Total number of states is therefore 3200×3200.

Problem 2.13

(i) Normal ordering ambiguities arise only when $m = -n$ (for L_m and L_n in l.h.s. they doesn't matter because constants will commute with operators):

$$[L_m, L_{-m}] = 2mL_0 + A(m),$$

and therefore redefinition $L_0 \to L_0 + C$ due to normal ordering ambiguities in quantum L_0 operator definition leads to redefinition $A(1) \to A(1) - 2C$. This lets us to set $A(1)$ to zero.

(ii) This is $sl(2,R)$ algebra:

$$[L_1, L_{-1}] = 2L_0, \quad [L_{\pm 1}, L_0] = \pm L_{\pm 1}.$$

Indeed, replacing $L_1 = e_-$, $L_{-1} = -e_+$ and $L_0 = H$ we will get commutation relations of $sl(2,R)$:

$$[e_+, e_-] = 2H, \quad [H, e_\pm] = \pm e_\pm.$$

Problem 2.14

We impose Jacobi identity on Virasoro operators L_m to guarantee they form an algebra. After substitution of general central extension

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0} \tag{2.16}$$

into Jacobi identity we will result in constraint

$$(m-n)A(m+n) + (n-p)A(n+p) + (p-m)A(p+m) = 0. \tag{2.17}$$

From (2.16) it obviously follows that $A(0) = 0$. For $p = -1,\ n = -m + 1$ (remember that in solution of Problem 2.13 we adjusted $A(1) = 0$) we get from (2.17)

$$(2 - m)A(-m) = (m + 1)A(m - 1), \tag{2.18}$$

from which it follows that $A(-1) = 0$. With respect to this we can also set $p = 1,\ n = -1 - m$ in (2.17) and get

$$(m + 2)A(-m) = (1 - m)A(m + 1). \tag{2.19}$$

Combining (2.18) and (2.19) we get a condition

$$A(m)(m - 2)(m - 3) = m(m + 1)A(m - 2),$$

that can be rewritten as

$$A(m) = \frac{m(m^2 - 1)}{(m - 2)\left((m - 2)^2 - 1\right)} A(m - 2).$$

This obviously means that

$$A(m) = Nm(m^2 - 1).$$

Constant N may be fixed if, e.g. $A(2)$ is known, therefore

$$A(m) = m(m^2 - 1)\frac{A(2)}{6}.$$

Such a result is purely quantum. It means that it's all due to normal-ordering ambiguity in L_0, which let us to add a constant to L_0 to make $A(1) = 0$ and result in (2.18), and so on. If we treat classical theory with algebra, determined by Poisson brackets, we have $[L_m, L_n]_{P.B.} = i(m - n)L_{m+n}$ with no central charges as a result of 1) clear-stated definition of L_0 with no additional terms ambiguities; 2) the fact that α do commute with each other: $\alpha^\mu_m \alpha^\nu_n = \alpha^\nu_n \alpha^\mu_m$, which has no deal with non-zero Poisson bracket.

3 Conformal field theory and string interactions

`Problem` **3.1**
Consider infinitesimal translation and special conformal transformation:

$$\delta_1 x^\mu = a^\mu, \quad \delta_2 x^\mu = b^\mu x^2 - 2x^\mu b \cdot x.$$

Lie bracket of this two transformations is equal to

$$(\delta_1\delta_2 - \delta_2\delta_1)x^\mu = \omega^\mu_\nu x^\nu + \lambda x^\mu,$$

where $\omega^\mu_\nu = 2(b^\mu a_\nu - a^\mu b_\nu),\ \lambda = 2a \cdot b$ are parameters of Lorentz rotation and scaling dilation transformations.

Problem 3.2

We deal with $SL(2,R)\times SL(2,R)$ generators $(l_0,\ l_1,\ l_{-1})\bigcup(\bar l_0,\ \bar l_1,\ \bar l_{-1})$. The following classification takes place:

$$l_{-1}=-\partial_z,\ \bar l_{-1}=-\partial_{\bar z}\quad \text{translations}$$
$$l_0+\bar l_0=-z\partial_z-\bar z\partial_{\bar z}\quad \text{dilations}$$
$$i(l_0-\bar l_0)=i(\bar z\partial_{\bar z}-z\partial_z)\quad \text{rotations}$$

The last generator actually performs transformation of the sort: $z\to -i\phi z$, which is definetely an infinetisamal rotation $z\to e^{-i\phi}z$. Special conformal transformations are generated by operators $l_1=-z^2\partial_z,\ \bar l_1=-\bar z^2\partial_{\bar z}$ and look like $z\to az+bz^2$. Therefore we've covered the whole $D=2$ conformal group with generic transformations of the form of BBS (3.7).

Problem 3.3

The idea of the solution is as follows: conformal group in D dimensions has $\frac12(D+1)(D+2)$ parameters, which is the same as for $SO(2,D)$. Conformal group consists of Lorentz transformations ($\frac12 D(D-1)$ generators $M_{\mu\nu}$), Poincare translations (D generators P_μ), special conformal transformations (D generators K_μ) and dilations (generator D). In the solution of problem 3.1 it was studied Lie bracket for special conformal transformation and translation, which may be used to find commutator of corresponding generators. Similarly, e.g., studying Lie bracket of special conformal transformation and Lorentz rotation we find commutator

$$[M_{\mu\nu},K_\rho]=-i(\eta_{\mu\rho}K_\nu-\eta_{\nu\rho}K_\mu).$$

Or we can actually simply use the procedure of commuting of operators using their explicit form, e.g., $K_\nu=i(x^2\partial_\nu-2\eta_{\mu\nu}x^\mu)$. As a result we will get commutators:

$$[M_{\mu\nu},D]=0,\quad [D,K_\mu]=iK_\mu,\quad [D,P_\mu]=-iP_\mu,\quad [P_\mu,K_\nu]=2iM_{\mu\nu}-2i\eta_{\mu\nu}D.$$

These commutators may be accompanied with Poincare algebra commutators in D dimensions.

If one defines generators J_{MN}

$$J_{\mu\nu}=M_{\mu\nu},\quad J_{\mu D}=\frac12(K_\mu-P_\mu),\quad J_{\mu(D+1)}=\frac12(K_\mu+P_\mu),J_{(D+1)D}=D,$$

with M,N being $D+1$-valued indices, then they will satisfy commutation relations of $SO(2,D)$.

Problem 3.4

OPE of some field and stress-energy tensor may be derived from the requiring the proper transformation law for this field under conformal transformation. It becomes especially easy if the field is primary. If we know the conformal weights of primary field we can determine OPE of this field and SET (see BBS (3.33)):

$$T(z)\Phi(w,\bar w)=\frac{h}{(z-w)^2}\Phi(w,\bar w)+\frac{1}{z-w}\partial\Phi(w,\bar w)+\dots. \tag{3.20}$$

Here $\Phi(z)=X(z)$ (we deal with right-movers, see BBS (3.19)) and $T(z)=-2:\partial X\cdot\partial X:$ (see BBS (3.23)). To start calculate OPE in this case we should know Green function for X beforehand (see BBS (3.35)):

$$\langle X_\mu(z,\bar z)X_\nu(w,\bar w)\rangle=-\frac14\eta_{\mu\nu}\left(\ln(z-w)+\ln(\bar z-\bar w)\right). \tag{3.21}$$

From (3.21) we can derive the following OPE:

$$\langle X_\mu(z)\partial X_\nu(w)\rangle = \frac{1}{4}\eta_{\mu\nu}\frac{1}{z-w}.$$

Using Wick theorem we can now calculate

$$T(z)X^\mu(w) = -2 : \partial X\cdot\partial X X^\mu : -4\partial X^\nu(z)\langle\partial X_\nu(z)X^\mu(w)\rangle = \frac{\partial X^\mu(z)}{z-w} + \cdots =$$

$$= \frac{\partial X^\mu(w)}{z-w} + \cdots .$$

The last transition is made due to the fact that

$$\frac{\partial X^\mu(z)}{z-w} = \frac{\partial X^\mu(w) + (z-w)\partial^2 X^\mu}{z-w} = \frac{\partial X^\mu(w)}{z-w} + \cdots .$$

Therefore we can conclude that $h = 0$ is a conformal dimension of primary field X^μ.

`Problem` **3.5**

(i) We can obtain required OPE's simply by $w, \bar{w}$-differentiation of expression

$$T(z)X^\mu(w) = \frac{\partial X^\mu(w)}{z-w} + \cdots .$$

obtained in the solution of Problem 3.4. Consequently

$$T(z)\partial X^\mu(w) = \frac{\partial X^\mu(w)}{(z-w)^2} + \frac{\partial^2 X^\mu(w)}{z-w} + \cdots ,$$

$$T(z)\bar{\partial} X^\mu(w) = \frac{\partial\bar{\partial} X^\mu(w,\bar{w})}{z-w} + \cdots = \cdots ,$$

$$T(z)\partial^2 X^\mu(w) = \frac{2\partial X^\mu(w)}{(z-w)^3} + \frac{2\partial^2 X^\mu(w)}{(z-w)^2} + \frac{\partial^3 X^\mu(w)}{z-w} + \cdots .$$

In the second raw we've used the fact that $X^\mu(w,\bar{w}) = X^\mu_L(\bar{w}) + X^\mu_R(w)$. But we can instead of it calculate

$$\tilde{T}(\bar{z})X^\mu(w,\bar{w}) = \frac{\bar{\partial} X^\mu(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial^2 X^\mu(\bar{w})}{\bar{z}-\bar{w}} + \cdots .$$

(ii) From these equations we can conclude that h-conformal dimension of ∂X^μ is equal to 1, for $\partial^2 X^\mu$ it's 2 and for $\bar{\partial} X^\mu$ it's 0 (while $\tilde{h} = 1$). Observe that basing on OPE's which are required to calculate in the condition of first part of the problem we wouldn't be able to conclude anything about $\tilde{h}$-conformal dimension of any of these fields because that is to be determined not from (3.20)-type OPE, but from $T(\bar{z})\Phi(\bar{w})$ one (as was done for calculation of $\tilde{h} = 1$ for $\bar{\partial} X^\mu$).

`Problem` **3.6**

1) We will use the following expression for singular part of the product

$$\partial X_\mu(z)\partial X_\nu(w) = -\frac{1}{4}\eta_{\mu\nu}\frac{1}{(z-w)^2}.$$

At the same time from

$$\partial X^\mu(z) = -\frac{i}{2}\sum_n \alpha_n^\mu z^{-n-1}$$

it follows that

$$\alpha_m^\mu = \frac{1}{\pi}\oint dz z^m \partial X^\mu(z),$$

where integration is performed over some contour around $z = 0$ pole point. Therefore

$$[\alpha_m^\mu, \alpha_n^\nu] = \frac{1}{\pi^2}\oint\oint dz dw z^m w^n \left(\partial X^\mu(z)\partial X^\nu(w) - \partial X^\nu(w)\partial X^\mu(z)\right).$$

In the last expression integral we shall take into account radius-ordering procedure in CFT. Therefore while performing integration over w first we should accomplish it over contour C around $w = z$ point. Second, we perform z integration over contour with $z = 0$ center. Using Cauchy's theorem we will get

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}.$$

2) Using the same technic we may derive the commutator for oscillator amplitudes of right-movers:

$$[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}.$$

3) From (3.21) it follows that

$$\partial X_\mu(z)\bar{\partial}X_\nu(\bar{w}) = 0.$$

Therefore we can easily conclude that

$$[\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0.$$

`Problem` **3.7**

We deal with primary field $\Phi(z)$ with conformal dimension h. The later condition allows us to write down the Loran series in the form

$$\Phi(z) = \sum_n \frac{\Phi_n}{z^{n+h}}.$$

From this it follows that

$$\Phi_n = \frac{1}{2\pi i}\oint dz \Phi(z) z^{n+h-1}, \quad \frac{1}{2\pi i}\oint dz \partial\Phi(z) z^{n+h} = -(n+h)\Phi_n. \tag{3.22}$$

Together with

$$L_m = \frac{1}{2\pi i}\oint dw T(w) w^{m+1}$$

it gives us

$$[L_m, \Phi_n] = \frac{1}{2\pi i}\oint dz z^{n+h-1}\frac{1}{2\pi i}\oint_C dw T(w)\Phi(z) w^{m+1},$$

where again C is some contour of w around z. Using an expression (3.20) and Cauchy's theorem we will result in

$$[L_m, \Phi_n] = \frac{1}{2\pi i}\oint dz z^{n+h-1}\left(h(m+1)z^m\Phi(z) + z^{m+1}\partial\Phi(z)\right) = -(n + m(1-h))\Phi_{n+m}, \tag{3.23}$$

where we've used (3.22) in the last transition.

`Problem` **3.8**

In a physical system with primary field $\Phi(z)$ with conformal dimension h vacuum $|0\rangle$ satisfies

$$L_n|0\rangle = 0,\ n > 0, \quad L_0|0\rangle = 0, \tag{3.24}$$

which is a special case of physical string state condition. Using (3.23), (3.24) we will get

$$L_0|\Phi\rangle = h|\Phi\rangle, \quad L_n|\Phi\rangle = 0,\ n > 0$$

for state $\Phi_{-h}|0\rangle$. Then this is a highest-weight state.

`Problem` **3.9**

(i) We can calculate 2-point correlation function for arbitrary primary fields $\phi_i(z_1, \bar{z}_1),\ \phi_j(z_2,\ \bar{z}_2)$ with conformal weights $(h_i,\ \tilde{h}_i),\ (h_j,\ \tilde{h}_j)$:

$$\langle \phi_i(z_1, \bar{z}_1)\phi_j(z_2,\ \bar{z}_2)\rangle_0 = \langle 0|\phi_i(z_1, \bar{z}_1)\phi_j(z_2,\ \bar{z}_2)|0\rangle.$$

This may be done due to the following considerations. Quantum field theory possessing conformal symmetry should have conformally invariant correlation functions. Therefore it should be

$$\delta_{\varepsilon,\bar{\varepsilon}}\langle 0|\phi_i(z_1, \bar{z}_1)\phi_j(z_2,\ \bar{z}_2)|0\rangle = 0, \tag{3.25}$$

where $\varepsilon(z)$ is a parameter of conformal transformation $z \to z + \varepsilon(z)$. We can find out for which types of correlation functions (3.25) holds invariant under action of $SO(2,2) = SL(2,R) \times SL(2,R) = (L_0,\ L_\pm)\bigcup(\tilde{L}_0,\ \tilde{L}_\pm)$ conformal transformations. Under the later transformations we know that vacuum $|0\rangle$ is invariant - it's an eigenstate with zero eigenvalue for $(L_0,\ L_\pm)$ operators, which follows from Virasoro algebra commutation rule

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1). \tag{3.26}$$

Physical vacuum is defined to possess $L_m|0\rangle = 0,\ m \geq 0$. If we apply (3.26) for $m = -n = 1$, we will immediately get $L_{-1}|0\rangle = 0$ (and similarly for left-moving generators $\tilde{L}$).

Therefore variation of correlation function (3.25) under $SO(2,2)$ conformal transformations is all due to variation of conformal fields:

$$\langle \delta_{\varepsilon,\bar{\varepsilon}}\phi_i(z_1, \bar{z}_1)\phi_j(z_2,\ \bar{z}_2)\rangle_0 + \langle \phi_i(z_1, \bar{z}_1)\delta_{\varepsilon,\bar{\varepsilon}}\phi_j(z_2,\ \bar{z}_2)\rangle_0 = 0.$$

We know how conformal transformation acts on primary field $\phi(z, \bar{z})$:

$$\delta_{\varepsilon,\bar{\varepsilon}}\phi(z,\bar{z}) = (h\partial\varepsilon(z) + \varepsilon(z)\partial + \tilde{h}\bar{\partial}\bar{\varepsilon}(\bar{z}) + \bar{\varepsilon}(\bar{z})\bar{\partial})\phi(z,\bar{z}). \tag{3.27}$$

Define for shortness of writing

$$G^{(2)}(z_i, \bar{z}_i) = \langle \phi_i(z_1, \bar{z}_1)\phi_j(z_2,\ \bar{z}_2)\rangle_0.$$

1) Translations $\varepsilon(z) = 1$ (or any small constant)

$$[(\partial_1 + \partial_2) + (\bar{\partial}_1 + \bar{\partial}_2)]G^{(2)}(z_i, \bar{z}_i) = 0,$$

which divides into two complex conjugate conditions

$$(\partial_1+\partial_2)G^{(2)}(z_i,\bar z_i)=0,\quad (\bar\partial_1+\bar\partial_2)G^{(2)}(z_i,\bar z_i)=0.$$

These conditions give us generic dependence $G^{(2)}(z_i,\bar z_i)=G^{(2)}(z_1-z_2,\bar z_1-\bar z_2)$.
2) Scaling $\varepsilon(z)=z$

$$[z_1\partial_1+z_2\partial_2+h_i+h_j+(c.c)]G^{(2)}(z_1-z_2,\bar z_1-\bar z_2)=0.$$

It follows for non-zero conformal weights that $G^{(2)}(z_1-z_2,\bar z_1-\bar z_2)=C_{ij}(z_1-z_2)^{-h_i-h_j}(\bar z_1-\bar z_2)^{-\tilde h_i-\tilde h_j}$.
3) Special conformal transformations $\varepsilon(z)=z^2$

$$[z_1^2\partial_1+z_2^2\partial_2+2h_iz_1+2h_jz_2+(c.c)]C_{ij}(z_1-z_2)^{-h_i-h_j}(\bar z_1-\bar z_2)^{-\tilde h_i-\tilde h_j}=0.$$

It follows that $h_i=h_j,\ \tilde h_i=\tilde h_j$ as a requirement, which is of course not true for different primary fields ϕ_i and ϕ_j, generally speaking. Therefore (up to normaliztion constant multiplier) it holds

$$\langle\phi_i(z_1,\bar z_1)\phi_j(z_2,\ \bar z_2)\rangle_0=\delta_{ij}\frac{1}{(z_1-z_2)^{2h_i}(\bar z_1-\bar z_2)^{2\tilde h_i}}.$$

In the case of zero conformal weight steps 2) and 3) are modified to give logarithmic correlation function. The simplest way to see this is to observe that derivative of zero-weight field is 1-weight field with known correlation function, integration of which gives us logarithm.
(ii) Again, 3-point correlation function should be invariant under $SL(2,2)$ conformal transformations, which restricts it in a manner of (i) to the view

$$G^{(3)}(z_i,\bar z_i)=\langle\phi_i(z_1,\bar z_1)\phi_j(z_2,\bar z_2)\phi_k(z_3,\bar z_3)\rangle_0=$$
$$=\frac{C_{ijk}}{z_{12}^{h_i+h_j-h_k}z_{13}^{h_i+h_k-h_j}z_{23}^{h_j+h_k-h_i}\bar z_{12}^{\tilde h_i+\tilde h_j-\tilde h_k}\bar z_{13}^{\tilde h_i+\tilde h_k-\tilde h_j}\bar z_{23}^{\tilde h_j+\tilde h_k-\tilde h_i}},$$

where $z_{ij}=z_i-z_j,\ \bar z_{ij}=\bar z_i-\bar z_j$. The constant of proportionality may be determined for $z_1=\infty,\ z_2=0,\ z_3=1$ as

$$C_{ijk}=\lim_{z_1\to\infty}z_1^{2h_i}\bar z_1^{2\tilde h_i}G^{(3)}.$$

This is a single constant which 3-point correlation function depends on.

`Problem` **3.10**
(i) According to Virasoro algebra commutation rule (3.26) and the constraint $L_0|\phi\rangle=h|\phi\rangle$ we find that

$$\langle\phi|[L_n,L_{-n}]|\phi\rangle=(2nh+\frac{c}{12}n(n^2-1))\langle\phi|\phi\rangle.$$

We also have the following constraint: $L_n|\phi\rangle=0,\ n>0$. Therefore (due to $L_{-n}^\dagger=L_n$) we proceed to

$$\langle\phi|L_{-n}^\dagger L_{-n}|\phi\rangle=(2nh+\frac{c}{12}n(n^2-1))\langle\phi|\phi\rangle.$$

This is a relation between positive norms of vectors $|\phi\rangle$ and $L_{-n}|\phi\rangle$ (details are explained bellow). For $n\to+\infty$ we get

$$\langle\phi|L_{-n}^\dagger L_{-n}|\phi\rangle\sim\frac{c}{12}n^3\langle\phi|\phi\rangle,$$

from which it follows that $c > 0$. For $n = 1$ we have

$$\langle\phi|L_{-1}^{\dagger}L_{-1}|\phi\rangle = 2h\langle\phi|\phi\rangle,$$

and therefore $h \geq 0$. For $h = 0$ we have zero-norm state $L_{-1}|\phi\rangle = 0$. See details in the next part of this solution

Here we also may add that $c = 0$ together with $h = 0$ would give zero norm for descendant states of the kind $L_{-n}|\phi\rangle$, which is impossible because we know that they are physical.

(ii) If $|\phi\rangle = |0\rangle$, then $L_0|\phi\rangle = 0$ due to constraints on conformal vacuum $L_n|0\rangle = 0,\ n \geq 0$. Then from constraint on highest weight state $L_0|\phi\rangle = h|\phi\rangle$ it follows $h = 0$.

If $h = 0$, we got $\langle\phi|L_{-1}^{\dagger}L_{-1}|\phi\rangle = 0$, therefore in a unitary Hilbert space $L_{-1}|\phi\rangle = 0$. Highest weight state $|\phi\rangle$ with zero conformal weight is determined by constraints $L_n|\phi\rangle = 0,\ n \geq 0$. These are vacuum-type constraints (as a consequence of Virasoro algebra this is $SL(2,R)$-invariance of vacuum), therefore $|\phi\rangle = |0\rangle$, and thus $L_{-1}|\phi\rangle = 0$.

`Problem` **3.11**

Consider the following expression for ghost SET:

$$T_{bc}(z) = -\lambda : b(z)\partial c(z) : +\varepsilon(\lambda - 1) : c(z)\partial b(z) : \ . \tag{3.28}$$

Here for fermi ghosts $\varepsilon = +1$ (if fields $b,\ c$ satisfy bose-statistics then $\varepsilon = -1$). From this formula and correlation functions for ghost conformal fields

$$\langle c(z)b(w)\rangle_0 = \frac{1}{z - w},$$

$$\langle b(z)c(w)\rangle_0 = \frac{\varepsilon}{z - w}$$

we can derive OPE of SET (3.28) with itself:

$$T_{bc}(z)T_{bc}(w) = \frac{c/2}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w},$$

where

$$c(\varepsilon,\ \lambda) = -2\varepsilon(6\lambda^2 - 6\lambda + 1).$$

The conformal anomaly term arised from completely contracted terms in Wick expansion of $T(z)T(w)$:

$$\lambda^2\langle b(z)\partial c(w)\rangle_0\langle\partial c(z)b(w)\rangle_0 + \lambda(\lambda - 1)\langle b(z)c(w)\rangle_0\langle\partial c(z)\partial b(w)\rangle_0 +$$

$$+\lambda(\lambda - 1)\langle c(z)b(w)\rangle_0\langle\partial b(z)\partial c(w)\rangle_0 + (\lambda - 1)^2\langle c(z)\partial b(w)\rangle_0\langle\partial b(z)c(w)\rangle_0.$$

`Problem` **3.12**

Our objective here is to draw out a simplest strategy of proving the nilpotency of operator

$$Q_B = \oint dz(cT_X + : bc\partial c :)$$

by anticommuting it with itself. We perform integration over two circle contours in double complex plane with coordinates z and w. To perform calculations we use the following key points:
1) Wick theorem with respect to minus sign arising when fermionic vacuum average (contraction, or correlation function) is brought out of normal product;
2) $T_X(z)T_X(w)$ OPE;
3) Ghost fileds OPE;
4) $:c(z)c(w):=-:c(w)c(z):$ and so on;
5) $\oint\oint dzdwf(z,w):c(z)c(w):=0$ for $f(z,w)=f(w,z)$ and so on;
6) $\oint dw:\partial^3c(w)c(w):=3!\oint\oint dwdz:c(z)c(w):\frac{1}{(z-w)^4}=0$;
7) $\oint\oint dzdw:A_1(z)A_2(w)F_1(z)F_2(w):=0$ for fermions $F_1,\ F_2$ and bosons $A_1,\ A_2$;
8) Cauchy's theorem to integrate around pole special points in $z=w$ terms over z.

Example of applying the point 7) arises while computation of Q_B^2:

$$\oint\oint dzdw:T_X(z)c(z)b(w)c(w)\partial c(w):=0,$$

where $T_X(z)$ together with, e.g., $b(w)c(w)$ are bosons.

The proof of point 7) is pretty obvious: double z,w integration is symmetric under interchange of variables $z\leftrightarrow w$, but fermions $F_1(z),\ F_2(w)$ are antisymetric. Therefore their contraction gives zero value, which already has shown itself in point 5). In point 7) presence of bosons changes nothing because they are permutable under normal product.

Point 8) is useful to integrate remaining (after applying 5) and 7) points) terms of Q_B^2 expression over z.

All this key points should be applied to straightforward Q_B^2 computation, which will result in $Q_B^2=0$.

An importatnt thing here to note is that there's no necessity for $c_X=0$ as a condition for $Q_B^2=0$. It does not contradict to the known fact that for $Q_B^2=0$ it's necessary to be $c=0$ anomaly of SET, because later deals with total SET anomaly term but not just T_X anomaly term c_X. Note also that in this problem we implicitly assumed total anomaly to be zero, because of considered theory deals with proper T_{bc} and corresponding proper ghost Lagrangian, therefore corresponding proper BRST symmetry transformations and BRST charge.

Problem 3.13

(i) Vertex operator has the form

$$V=:V_RV_L:,$$

where

$$V_R=\zeta_R\cdot\partial X_R(z)\exp\left(ik\cdot X_R(z)\right),\quad V_L=\zeta_L\cdot\partial X_L(\bar z)\exp\left(ik\cdot X_L(\bar z)\right),$$

for bosonic string state of the type $(\zeta_R\cdot\alpha_{-1}+\zeta_L\cdot\tilde\alpha_{-1})|0;k\rangle$ (with $N_L=N_R=1$). Such a form of vertex operator is chosen to enable $\alpha_{-1}\tilde\alpha_{-1}|0\rangle$ form of the state.

(ii) Because of X_R^μ being a primary (conformal) field with conformal weights $(h,\tilde h)=(0,0)$, vertex operator V_R is a primary field with conformal weights $(h,\tilde h)=(1,0)$, where h is composed out of $\frac{k^2}{8}$ from $\exp\left(ik\cdot X_R(z)\right)$ and 1 from $\partial X_R(z)$, which gives $h=1$ in the sum due to the mass-shell condition $k^2=0$, imposed independently. The physical state associated with primary field with conformal dimension h is built as $|\phi\rangle=V_{-h}|0\rangle$, where V_{-h} is to be taken from expansion

$$V_R(z) = \sum_n \frac{V_n}{z^{n+h}},$$

and a vacuum state $|0\rangle$ satisfies constraints $V_n|0\rangle = 0,\ n > -h$. The same arguments are hold for left-movers. Therefore the state, corresponding to primary field $V(z,\bar{z})$ looks like

$$|\phi\rangle = V_{(R)-1}V_{(L)-1}|0\rangle.$$

In the solution of Problem 3.8 it was shown that the state built out of primary field in a such manner satisfies physical state conditions.

`Problem` **3.14**

We start with a little introduction. The spectrum of bosonic string in BRST quantization approach is a spectrum of bosonic excitations and ghosts excitations. Each physical state belongs to some level, i.e., each physical state has some level number. Mass-shell condition is equivalent to zero-mode Virasoro constraint of total SET:

$$L_0|\psi\rangle = 0 \to \alpha'(p^2 + m^2)|\psi\rangle = 0,$$

where

$$\alpha' m^2 = \sum_{n>0} n(N_{bn} + N_{cn} + \tilde{N}_{\tilde{b}n} + \tilde{N}_{\tilde{c}n}) + N + \tilde{N} - 1, \tag{3.29}$$

and we've used usual bosonic number operators

$$N = \sum_{n>0} \alpha_{-n} \cdot \alpha_n, \qquad \tilde{N} = \sum_{n>0} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n,$$

and ghost number operators for each excitation level

$$N_{bn} = c_{-n}b_n, \quad N_{cn} = b_{-n}c_n, \quad \tilde{N}_{\tilde{b}n} = \tilde{c}_{-n}\tilde{b}_n, \quad \tilde{N}_{\tilde{c}n} = \tilde{b}_{-n}\tilde{c}_n. \tag{3.30}$$

The total number operator of bosonic string BRST spectrum is

$$N_{tot} = \alpha' m^2 + 1.$$

This operator commutes with BRST charge, because it's equal to the number of ghosts plus the number of antighosts, consequently while Q_B increases the number of antighosts by 1 it decreases the number of ghosts by 1, leaving N_{tot} constant. The presence of n-multiplier in the summation over n in (3.29) when we insert (3.30) into it has clear meaning, e.g., it counts that $c_{-2}|0;k\rangle$ corresponds to $+2$ antighost number, because ghost anticommutator $\{c_n, b_{-m}\} = \delta_{m,n}$ has no m-multiplier of Cronecer symbol on the r.h.s. in difference with α-commutator case. An examples useful for the construction of first (zero-mass) level:

$$N_{b1}c_{-1}|0;k\rangle = c_{-1}|0;k\rangle, \quad N_{c1}c_{-1}|0;k\rangle = 0, \quad N_{b1}b_{-1}|0;k\rangle = 0, \quad N_{c1}b_{-1}|0;k\rangle = b_{-1}|0;k\rangle.$$

1) For BRST invariance condition on physical states $Q_B|\phi\rangle = \tilde{Q}_B|\phi\rangle = 0$ (which is the condition $Q_B|\phi\rangle + \tilde{Q}_B|\phi\rangle = 0$ separated into independent parts) to be able to retrieve Virasoro physical state conditions on vacuum string state and excited states we should also impose

Siegel constraint $b_0|\phi\rangle = 0$ and no-ghost constraints $b_n|\phi\rangle = 0,\ n > 0$ (and all corresponding $\tilde{b}$ constraints).

Hilbert space of string states decomposes into ghost states and α-excitation states (we assume $\tilde{\alpha}$-excitations for closed string too) of bosonic coordinate fields X^μ as a direct product. Therefore we can build ghost spectrum and α-spectrum separately, and then identify level excitation number via ghost number operator U and string number operators $N,\ \tilde{N}$. On the same level $N = \tilde{N}$ due to level-matching condition.

Vacuum is identified as a state annihilated by lowering operators either for ghosts or for α-excitations. Lowering operators are the ones with positive index. Ghosts zero-indices operators are supposed to annihilate vacuum state too, because in difference to α_0 they depict no-momentum state. But they can't do it simultaneously, because of their non-zero anticommutator. Therefore there're two vacuum states: annihilated by either b_0 or c_0. As it's mentioned above for vacuum state to satisfy Virasoro constraint $L_0|0;k\rangle = |0;k\rangle$ it's necessary to impose namely b_0-type constraint. Therefore for vacuum string state we have conditions for right-movers

$$b_n|0;k\rangle = 0, \quad c_n|0;k\rangle = 0, \quad \alpha_n|0;k\rangle = 0,\ n > 0$$

for left-movers

$$\tilde{b}_n|0;k\rangle = 0, \quad \tilde{c}_n|0;k\rangle = 0, \quad \tilde{\alpha}_n|0;k\rangle = 0,\ n > 0$$

and Siegel constraints

$$b_0|0;k\rangle = 0, \quad \tilde{b}_0|0;k\rangle = 0.$$

The excitation numbers are $N = \tilde{N} = 0,\ U = -\frac{1}{2}$ (ghost number operator is defined as $U = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty}(c_{-n}b_n - b_{-n}c_n)$, see BBS (3.89)).

There're no BRST exact vacuum states $|\rangle = Q_B|\chi\rangle$, because BRST operator Q_B raises ghost number by 1 (remind that ghost number of b-field is -1, and ghost number of c-field is +1), therefore ghost number of $|\chi\rangle$ should be by 1 lower than ghost number of $|0;k\rangle$ vacuum state, i.e., it should be $|\chi\rangle = b_n|0;k\rangle$ or so. And then our BRST-exact state is (here it's necessary that $n < 0$ for b_n not to annihilate ghost vacuum) to

$$|\rangle = Q_B b_n|0;k\rangle = \{Q_B, b_n\}|0;k\rangle = (L_n - \delta_{n,0})|0;k\rangle = 0, \tag{3.31}$$

because of full Virasoro algebra is free of conformal anomaly and physical state condition on vacuum may be written as $(L_n - \delta_{n,0})|0;k\rangle = 0$ for all n.

2) Now let's study first excited level of bosonic string in BRST quantization. Begin with right-movers. We should impose a BRST quantization prescription

$$Q_B|\psi\rangle = 0. \tag{3.32}$$

and a Siegel constraint

$$b_0|\psi\rangle = 0.$$

We will use an expansion of BRST charge

$$Q_B = \sum_m (L_{-m}^{(X)} - \delta_{m,0})c_m - \frac{1}{2}\sum_m (m-n) : c_{-m}c_{-n}b_{m+n} : . \tag{3.33}$$

Combination of a Sigel constraint with (3.32) will result in a a mass-shell condition

$$(L_0 - 1)|\psi\rangle = 0,$$

where normal-ordered zero mode of the full SET is (as pointed in the very beginning of this solution but with shift of L_0 for convenience of notation there and with ordinary notation without shift here)

$$L_0 = \sum_{n>0} \alpha_{-n}\alpha_n + \frac{1}{2}\alpha_0^2 + \sum_{n>0} n(c_{-n}b_n + b_{-n}c_n).$$

On the general first level state

$$|\psi\rangle = (e \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1})|0;k\rangle \tag{3.34}$$

with 28 degrees of freedom the mass-shell condition is realized as follows

$$0 = (L_0 - 1)|\psi\rangle = (L_0^X - 1)|\psi\rangle + (c_{-1}b_1 + b_{-1}c_1)(e \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1})|0;k\rangle =$$
$$= (L_0^X - 1)|\psi\rangle + (\gamma c_{-1} + \beta b_{-1})|0;k\rangle. \tag{3.35}$$

Now using (3.33) expansion we can write constraint (3.32) for general first level state (3.34):

$$0 = Q_B|\psi\rangle = c_0(L_0^X - 1)|\psi\rangle - \frac{1}{2}\sum_m (m-n) : c_{-m}c_{-n}b_{m+n} : (e \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1})|0;k\rangle +$$
$$+(c_1 L_{-1}^X + c_{-1}L_1^X)(e \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1})|0;k\rangle = c_0\left((L_0^X - 1)|\psi\rangle + (\gamma c_{-1} + \beta b_{-1})|0;k\rangle\right) +$$
$$+(c_{-1}\alpha_0 \cdot \alpha_1 + c_1\alpha_0 \cdot \alpha_{-1})|\psi\rangle = (\alpha_0 \cdot e c_{-1} + \beta\alpha_0 \cdot \alpha_{-1})|0;k\rangle.$$

In the last transition we've used a mass-shell constraint (3.35).

Obviously the norm of considered state is

$$\langle\psi|\psi\rangle = |e|^2\langle 0;k|0;k\rangle.$$

Here $|e|^2 = -|e_0|^2 + \sum_{i=1}^{25}|e_i|^2$.

Therefore we see that for physical state it's necessary $\beta = 0,\ p \cdot e = 0$, where we've used $\alpha_0 \sim p$ expression for string center-of-mass momentum. Therefore there're are 26 linearly independent states left: 24 of them are bosonic states with positive norm and 2 are antighost c_{-1} state and $p \cdot \alpha_{-1}$ zero-norm (because of $p^2 = 0$) state, this 2 states are orthogonal to all physical states including themselves (physical states with zero norm, orthogonal to all physical states, are null spurious states).

Now let's study cohomology group of considered first-level state $|\psi\rangle$. All BRST exact first-level states $|\chi\rangle$ should be built out of states $|\psi'\rangle$ with the general view (3.34), because operator Q_B obviously commutes with the number operator due to BRST invariance of a state. Therefore all BRST exact states have the form of

$$|\chi\rangle = (\alpha_0 \cdot e' c_{-1} + \beta'\alpha_0 \cdot \alpha_{-1})|0;k\rangle = 0.$$

Therefore states of ghost field $c_{-1}|0;k\rangle$ and longitudinal state $p \cdot \alpha_{-1}|0;k\rangle$ are unphysical, and may be changed inside the same cohomology class. We have therefore just 24 physical states of bosonic string on first level - the same result as obtained in light-cone quantization.

All contemplations applied for right-movers may be repeated for left movers just word by word with the change $\alpha \to \tilde{\alpha},\ b \to \tilde{b},\ c \to \tilde{c}$. As a result we will get $24^2 = 576$ physical states of the form

$$|\psi\rangle = (e \cdot \alpha_{-1} + \gamma c_{-1} + \tilde{e} \cdot \tilde{\alpha}_{-1} + \tilde{\gamma}\tilde{c}_{-1})|0;k\rangle$$

with the norm

$$\langle\psi|\psi\rangle = (|e|^2 + |\tilde{e}|^2)\langle 0;k|0;k\rangle.$$

States of ghost fields $c_{-1}|0;k\rangle, \tilde{c}_{-1}|0;k\rangle$ and longitudinal states $p\cdot\alpha_{-1}|0;k\rangle, p\cdot\tilde{\alpha}_{-1}|0;k\rangle$ are unphysical. As a result we've obtained transversal bosonic string spectrum but without breaking explicit covariance of approach.

Problem 3.15

Here $N_{tot} = 2$ for the state of an open string spectrum. The most general state is

$$|\phi\rangle = \left(\zeta\cdot\alpha_{-2} + \Omega_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1} + \beta b_{-2} + \gamma c_{-2} + \rho c_{-1}b_{-1} + c_{-1}\xi_1\cdot\alpha_{-1} + b_{-1}\xi_2\cdot\alpha_{-1}\right)|0;k\rangle.$$

Here only symmetric part $\Omega_{\mu\nu} = \Omega_{\nu\mu}$ of Ω-matrix survives, but when we will be considering construction of BRST-exact states bellow we will not restrict ourselves by such condition on Ω-matrix. The same view has the most general non-closed BRST state, on which action of Q_B gives $N_{tot} = 2$ BRST-exact physical state. The action of BRST charge (3.33) can be easily written as follows

$$Q_B|\phi\rangle = c_0(L_0-1)|\phi\rangle + (\beta\left(\alpha_{-2}\cdot\alpha_0 + \frac{1}{2}\alpha_{-1}\cdot\alpha_{-1}\right) + c_{-2}\zeta\cdot\alpha_0 + c_{-2}\Omega^{\mu}_{\mu} +$$

$$+2c_{-1}\Omega_{\mu\nu}\alpha^{\mu}_0\alpha^{\nu}_{-1} - \rho\alpha_{-1}\cdot\alpha_0 c_{-1} + 3\rho c_{-2} + \alpha_{-1}\cdot\alpha_0\xi_2\cdot\alpha_{-1} + \xi_2\cdot\alpha_{-2})|0;k\rangle, \qquad (3.36)$$

and we also have mass-shell condition

$$(L_0-1)|\phi\rangle = 0.$$

Notice that $c_{-1}\xi_1\cdot\alpha_{-1}|0;k\rangle$ states are BRST-exact (thanks to the term $c_{-1}\Omega_{\mu\nu}\alpha^{\mu}_0\alpha^{\nu}_{-1}$, which leads to equation of the form $\Omega_{\mu\nu}\alpha^{\mu}_0 = \xi_{1\nu}$) and we can right now exclude all such states and therefore parameters ξ_1 from our consideration (of course there're another BRST-exact states, which will be considered bellow). Therefore for the state $|\phi\rangle$ to be BRST-closed it's necessary (we put $\alpha^{\mu}_0 = p^{\mu}$) for the following conditions to be satisfied

$$\beta = 0, \quad \zeta\cdot p + \mathrm{tr}\Omega = 0, \quad \xi^{\mu}_2 = 0, \quad \Omega_{\mu\nu}\alpha^{\nu}_0 = 0, \quad \rho = 0.$$

BRST-exact states of $N_{tot} = 2$ are constructed with the help of operators (we just enumerate kinds of terms in (3.36) except for excluded $c_{-1}\xi_1\cdot\alpha_{-1}$)

$$\alpha_{-2}\cdot\alpha_0, \quad \alpha_{-1}\cdot\alpha_{-1}, \quad c_{-2}, \quad \alpha_{0\mu}\xi_{2\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}, \quad \xi_2\cdot\alpha_{-2}. \qquad (3.37)$$

Among them $\xi_2\cdot\alpha_{-2}$ may be already excluded by imposing a condition of BRST-closeness, because the idea is to exclude from BRST-closed states all which are BRST-exact too. We can calculate total number of states. Initially we had

$$\dim(|\phi\rangle) - \dim(\xi_1) = 26 + \frac{26\cdot 27}{2} + 3 + 26,$$

then we have closeness constraints of number 52+3. Finally among BRST-closed states there're $1+26$ BRST-exact (because among (3.37) first and second operators are particular cases of the fifth and fourth correspondingly and we've excluded $\xi_2\cdot\alpha_{-2}$ as mentioned above). As a result there're 324 physical states on $N_{tot} = 2$, which are not BRST-exact. The same number was obtained in light-cone quantization.

4 Strings with world-sheet supersymmetry

Problem 4.1

(0) First of all, fermionic Lagrangian describes D Majorana fermions each of which belongs to $d = 1$ fundamental representation of Cl_1 Clifford algebra. That's why there's no γ_5-type matrices and Dirac conjugate spinors in Lagrangian. Situation will change after quantization, see Problem 4.3.

(i) The Lagrangian considered in this problem is a special case of massless scalar field + massless fermionic field SUSY Lagrangian for 1D case. The bosonic part has clear physical meaning because of giving correct equation of motion for free particle (BTW, for massive particle too). Using Lagrange equations we can derive equations of motions

$$\ddot{X}^\mu = 0, \qquad \dot{\psi}^\mu = 0.$$

It was also used the fact of anticommutation of $\frac{\partial}{\partial\psi}$ with ψ while using Lagrange equation for fermionic field.

(ii) Here we should use $\psi\varepsilon = -\varepsilon\psi$ while calculating δS_0. Proper boundary conditions are $\psi^\mu \dot{X}_\mu = 0$ at $\tau = \pm\infty$, which are necessary because $\delta S_0 = \frac{i}{2}\varepsilon \int d\tau \frac{d}{d\tau}(\dot{X}_\mu \psi^\mu)$. General formula with auxiliary fields, valid for local SUSY transformations, is derived in (ii) point of the Problem 4.2

(iii) Majorana fermion ψ is real and stays real due to equations of motion. Under SUSY transformation $\delta X^\mu = i\varepsilon\psi^\mu$ bosonic coordinate field X^μ should vary by real value, therefore ε ought to be real. At the same time we have $(\varepsilon_1\varepsilon_2)^\star = \varepsilon_2\varepsilon_1$. We can find

$$(\delta_1\delta_2 - \delta_2\delta_1)X^\mu = \frac{i}{2}(\varepsilon_2\varepsilon_1 - \varepsilon_1\varepsilon_2)\dot{X}^\mu = \delta\tau \dot{X}^\mu,$$

where $(\delta\tau)^\star = \delta\tau$ due to pointed above. As expected Lie bracket of SUSY transformations on fermionic field also closes on translation:

$$(\delta_1\delta_2 - \delta_2\delta_1)\psi^\mu = \frac{i}{2}(\varepsilon_2\varepsilon_1 - \varepsilon_1\varepsilon_2)\dot{\psi}^\mu = \delta\tau\dot{\psi}^\mu.$$

Problem 4.2

(i) This may be proved in two ways. First is infinitesimal with the aid of direct application of given infinitesimal variations. The second way is based on the following facts: 1) Infinitesimal variations $\delta e = \frac{d}{d\tau}(\xi e)$ and $\delta\chi = \frac{d}{d\tau}(\xi\chi)$ for $\tau' = \tau - \xi(\tau)$ are small limits of full transformations $e(\tau)d\tau = e'(\tau')d\tau'$ and $\chi(\tau)d\tau = \chi'(\tau')d\tau'$; 2) if we write $X = X(\tau(\tau'))$ we will obtain $\frac{dX}{d\tau'} = \frac{dX}{d\tau}\frac{d\tau}{d\tau'}$, where both derivatives in the r.h.s. are taken in τ' time moment; point 2) can be rewritten literally for fermionic field ψ. Therefore we got

$$d\tau \left(\frac{\dot{X}^\mu \dot{X}_\mu}{2e} + \frac{i\dot{X}^\mu \psi_\mu \chi}{e} - i\psi^\mu \dot{\psi}_\mu \right) =$$

$$= d\tau' \frac{d\tau}{d\tau'} \left(\frac{1}{2e'(\tau')} \frac{d\tau}{d\tau'} \left(\frac{dX}{d\tau'}\right)^2 \left(\frac{d\tau'}{d\tau}\right)^2 + \frac{i}{e'(\tau')} \frac{d\tau}{d\tau'} \frac{dX^\mu}{d\tau'} \frac{d\tau'}{d\tau} \psi_\mu \chi'(\tau') \frac{d\tau'}{d\tau} - i\psi^\mu \frac{d\psi_\mu}{d\tau'} \frac{d\tau'}{d\tau} \right).$$

Obviously this ends the proof of invariance of action under reparametrization.
(ii) We should use anticommutation of classical fermions $\psi,\ \varepsilon,\ \chi$ and their derivatives; we will result in the following variation of action under local SUSY transformations:

$$\delta\tilde{S}_0 = \int d\tau \frac{d}{d\tau}\left(\frac{i}{2e}\varepsilon\psi_\mu \dot{X}^\mu\right).$$

(iii) Lagrange equations for auxiliary fields $\chi,\ e$ give rise to constraints

$$\frac{\dot{X}^\mu\psi_\mu}{e} = 0, \qquad \frac{\dot{X}^\mu\dot{X}_\mu}{2e^2} + \frac{\dot{X}^\mu\psi_\mu\chi}{e^2} = 0,$$

which admit the solution (gauge choice) $e = 1,\ \chi = 0$ and as a result corresponding constraints look like

$$\dot{X}^\mu\psi_\mu = 0, \qquad \dot{X}^2 = 0.$$

The second is the mass-shell condition.

`Problem` **4.3**
(i) Equations of motion for supersymmetric particle in Problem 4.1 are

$$\ddot{X}^\mu = 0, \qquad \dot{\psi}^\mu = 0.$$

General solution is

$$X^\mu(\tau) = x^\mu + p^\mu\tau, \qquad \psi^\mu = b^\mu,$$

where b^μ are D Majorana $d = 1$ constant in time fermions. Therefore canonical quantization requires $\{b^\mu,\ b^\nu\} = \eta^{\mu\nu}$ and $[x^\mu,\ p^\nu] = i\eta^{\mu\nu}$. Because of $P^\mu = \dot{X}^\mu = p^\mu$ and $[p^\mu,\ p^\nu] = 0$, the later means $[X^\mu,\ \dot{X}^\nu] = i\eta^{\mu\nu}$.
(ii) Ground state satisfies the condition

$$p^\mu|0\rangle = 0.$$

Fermionic part should be incorporated too. Because of absence of any fermionic excitations (fermions don't have time dependence), fermions b^μ should appear already in vacuum state $|0, b\rangle$. At the same time for $\Gamma^\mu = \frac{1}{\sqrt{2}}b^\mu$ from quantization condition we obtain Dirac matrices anticommutation relations

$$\{\Gamma^\mu,\ \Gamma^\nu\} = 2\eta^{\mu\nu},$$

which means that fermionic part of the vacuum forms a representation of Dirac algebra: $b^\mu|0;a\rangle = \frac{1}{\sqrt{2}}\Gamma^\mu_{ab}|0;b\rangle$. Here b is an index running $2^{[D/2]}$ values of dimension of fundamental representation of $Cl_{D-1,1}$ algebra. Therefore vacuum state is space-time fermion, which leads to the fact that excited states (for free particle excited states are non-zero momentum states) are space-time fermions too (fermions with non-zero momentum). They are built as $|k^\mu;b\rangle = k^\mu|0;b\rangle$:

$$p^\mu|k;b\rangle = k^\mu|k;b\rangle.$$

To cope with this one may consider $p^\mu = \int dk k^\mu|k\rangle\langle k|$ diagonal operator.
(iii) The constraint $\dot{X}^2 = 0$ means that our space-time fermion is massless. The constraint

$\dot{X} \cdot \psi = 0$ after quantization looks like $k_\mu \Gamma^\mu_{ab} |b; k\rangle = 0$. This is massless Dirac equation for $Cl_{D-1,1}$ spinors in momentum representation.

`Problem 4.4`

We consider *local* supersymmetry transformations

$$\delta X^\mu = \bar{\varepsilon}\psi^\mu, \quad \delta\psi^\mu = \rho^\beta(\partial_\beta X^\mu)\varepsilon + B^\mu\varepsilon, \quad \delta\bar{\psi}^\mu = -(\partial_\beta X^\mu)\bar{\varepsilon}\rho^\beta + B^\mu\bar{\varepsilon}, \quad \delta B^\mu = \bar{\varepsilon}\rho^\alpha\partial_\alpha\psi^\mu.$$

Variation of action

$$\tilde{S} = \int d^2\sigma \left[\partial_\alpha X^\mu \partial^\alpha X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu - B^\mu B_\mu\right]$$

is therefore

$$\delta\tilde{S} = \int d^2\sigma [2\partial_\alpha X^\mu \partial^\alpha \bar{\varepsilon}\psi_\mu - (\partial_\beta X^\mu)\bar{\varepsilon}\rho^\beta\rho^\alpha\partial_\alpha\psi_\mu + \bar{\psi}^\mu\rho^\alpha\partial_\alpha\rho^\beta(\partial_\beta X_\mu)\varepsilon +$$

$$+B^\mu\bar{\varepsilon}\rho^\alpha\partial_\alpha\psi_\mu + \bar{\psi}^\mu\rho^\alpha\partial_\alpha B_\mu\varepsilon - 2\bar{\varepsilon}\rho^\alpha(\partial_\alpha\psi_\mu)B^\mu].$$

Last three terms with the help of the following identity for Majorana fermions

$$\bar{\chi}\rho^\alpha\psi = -\bar{\psi}\rho^\alpha\chi$$

may be rewritten as

$$-\int d^2\sigma \partial_\alpha(B_\mu\bar{\varepsilon}\rho^\alpha\psi^\mu),$$

which will not contribute to supercurrent because of being an integral of total derivative. Notice that for Majorana fermions ψ, ε it also takes place

$$\bar{\psi}_\mu\rho^\alpha\rho^\beta\varepsilon = \bar{\varepsilon}\rho^\beta\rho^\alpha\psi_\mu,$$

which is not too difficult to prove. Applying this identity for our variation of action we will get

$$\delta\tilde{S} = \int d^2\sigma[2\partial_\alpha X^\mu\partial^\alpha\bar{\varepsilon}\psi_\mu - (\partial_\beta X^\mu)\bar{\varepsilon}\rho^\beta\rho^\alpha\partial_\alpha\psi_\mu + (\partial_\alpha((\partial_\beta X_\mu)\bar{\varepsilon}))\rho^\beta\rho^\alpha\psi^\mu - \partial_\alpha(B_\mu\bar{\varepsilon}\rho^\alpha\psi^\mu)] =$$

$$= \int d^2\sigma[\partial^\alpha(2(\partial_\alpha X^\mu)\bar{\varepsilon}\psi_\mu) - (\partial_\beta X^\mu)\bar{\varepsilon}\rho^\beta\rho^\beta\psi_\mu - B_\mu\bar{\varepsilon}\rho^\alpha\psi^\mu) - 2(\partial_\alpha\partial^\alpha X^\mu)\bar{\varepsilon}\psi_\mu + 2\partial_\alpha((\partial_\beta X^\mu)\bar{\varepsilon})\rho^\beta\rho^\alpha\psi_\mu)] =$$

$$= \int d^2\sigma[-2(\partial_\alpha\partial^\alpha X^\mu)\bar{\varepsilon}\psi_\mu + 2\partial_\alpha((\partial_\beta X^\mu)\bar{\varepsilon})(\eta^{\beta\alpha} + \frac{1}{2}(\rho^\beta\rho^\alpha - \rho^\alpha\rho^\beta))\psi_\mu)] + \cdots .$$

Dots represent terms with integration of total derivatives having no affection on supercurrent. We also antisymmetrize the product of Dirac matrices because then it will be convenient to perform summation of it with symmetric $\partial_\alpha\partial_\beta X^\mu$ being the term from $\partial_\alpha((\partial_\beta X^\mu)\bar{\varepsilon})$ (this summation gives zero result). As a result we obtain

$$\delta\tilde{S} = 2\int d^2\sigma(\partial_\beta X^\mu)(\partial_\alpha\bar{\varepsilon})\rho^\beta\rho^\alpha\psi_\mu.$$

The rest is obvious: two-component (spinor) supercurrent is

$$J^\alpha_A = 2(\rho^\beta\rho^\alpha\psi_\mu)_A\partial_\beta X^\mu.$$

Invariance of action (up to total derivative) requires convergence of cupercurrent. The later requires being on-shell. An important thing is that we haven't used equations of motion for the deduction of expression

$$\delta\tilde{S} = \int d^2\sigma J^\alpha \partial_\alpha \bar{\varepsilon}.$$

For global SUSY transformations action will be invariant up to a total derivative even off-shell (while convergence of supercurrent requires being on-shell). And this is held even if we deal with SUSY with no auxiliary fields, because later are of use to close of super-Poincare algebra off-shell, not for SUSY invariance.

`Problem` **4.5**

We deal with open string therefore $\alpha_0^\mu = l_s p^\mu$. At the same time we have $l_s = \sqrt{2\alpha'}$.

NS sector: zero-point Virasoro constraint with $a_{NS} = \frac{1}{2}$ is

$$L_0 - \frac{1}{2} = 0,$$

where normal ordered zero Virasoro operator is given by

$$L_0 = \frac{1}{2}\alpha_0^2 + N,$$

with number operator

$$N = \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{r>0} r b_{-r}^i b_r^i.$$

Therefore we get mass formula

$$\alpha' M^2 = N - \frac{1}{2}.$$

R sector: zero point Virasoro constraint is built with $a_R = 0$ and

$$L_0 = \frac{1}{2}\alpha_0^2 + N,$$

where number operator is

$$N = \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{n>0} n d_{-n}^i d_n^i.$$

Corresponding mass formula is

$$\alpha' M^2 = \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{n>0} n d_{-n}^i d_n^i.$$

`Problem` **4.6**

R sector for arbitrary dimension of space-time D in light-cone quantization:

$$L_0 = \frac{1}{2}\sum_n \sum_{i=1}^{D-2} \alpha_{-n}^i \alpha_n^i + \frac{1}{2}\sum_n \sum_{i=1}^{D-2} n d_{-n}^i d_n^i =$$

$$= \sum_{n>0}\sum_{i=1}^{D-2}\left(\alpha^i_{-n}\alpha^i_n + n d^i_{-n} d^i_n\right) + \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n>0}\sum_{i=1}^{D-2}\left([\alpha^i_n,\ \alpha^i_{-n}] - n\{d^i_n,\ d^i_{-n}\}\right) =$$

$$= \sum_{n>0}\sum_{i=1}^{D-2}\left(\alpha^i_{-n}\alpha^i_n + n d^i_{-n} d^i_n\right) + \frac{1}{2}\alpha_0^2.$$

It proves that $a_R = 0$.

NS sector for arbitrary D in light-cone quantization:

$$L_0 = \frac{1}{2}\sum_{n}\sum_{i=1}^{D-2}\alpha^i_{-n}\alpha^i_n + \frac{1}{2}\sum_{r}\sum_{i=1}^{D-2} r b^i_{-r} b^i_r =$$

$$= \sum_{n>0}\sum_{i=1}^{D-2}\alpha^i_{-n}\alpha^i_n + \sum_{r>0}\sum_{i=1}^{D-2} r b^i_{-r} b^i_r + \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n>0}\sum_{i=1}^{D-2}[\alpha^i_n,\ \alpha^i_{-n}] - \frac{1}{2}\sum_{r>0}\sum_{i=1}^{D-2} r\{b^i_r,\ b^i_{-r}\}.$$

That means

$$a_{NS} = -\frac{D-2}{2}\left(\sum_{n=1}^{\infty} n - \sum_{r=1/2}^{\infty} r\right) = \frac{D-2}{16},$$

for which we've used zeta-function regularization. In the case of $D = 10$ it gives $a_{NS} = \frac{1}{2}$. Let us clarify a thing about used zeta function regularization. Define $S_{even} = \sum_{n=0}^{\infty} 2n = 2\zeta(-1) = -\frac{1}{6}$ and $S_{odd} = \sum_{n=0}^{\infty}(2n+1)$. Then (we introduce one more redundant equation just out of curiosity of exploration of infinite sums)

$$S_{odd} + S_{even} = -\frac{1}{12}\,,$$

$$S_{odd} - S_{even} = \sum_{n=0}^{\infty} 1\,.$$

The second equation is the one which we do not use. From the first one we conclude $S_{odd} = \frac{1}{12}$. This is what we used above. Now we also can conclude

$$\sum_{n=0}^{\infty} 1 = \frac{1}{4}\,.$$

`Problem 4.7`

R sector The numbers pointed bellow should be multiplied by 8 - the number of spinor $|\psi_0\rangle$ components:

$d^i_{-2}|\psi_0\rangle$	8
$d^i_{-1}d^j_{-1}|\psi_0\rangle$	28
$\alpha^i_{-2}|\psi_0\rangle$	8
$\alpha^i_{-1}\alpha^j_{-1}|\psi_0\rangle$	36
$d^i_{-1}\alpha^j_{-1}|\psi_0\rangle$	64

Total number of states is $1152 = 144 \times 8$

NS sector The numbers in front of each state are the numbers of that states

$$b^i_{-5/2}|0\rangle \qquad 8$$

$$b^i_{-3/2}\alpha^j_{-1}|0\rangle \qquad 64$$

$$b^i_{-1/2}\alpha^j_{-2}|0\rangle \qquad 64$$

$$b^i_{-1/2}\alpha^j_{-1}\alpha^k_{-1}|0\rangle \qquad 288$$

$$b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}\alpha^l_{-1}|0\rangle \qquad 448$$

$$b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}b^l_{-1/2}b^m_{-1/2}|0\rangle \qquad 56$$

$$b^i_{-3/2}b^j_{-1/2}b^k_{-1/2}|0\rangle \qquad 224$$

Total number of states is 1152. Notice that numbers of bosonic and fermionic space-time states are equal to each other.

`Problem 4.8`

It may be verified directly (take into account that $\rho_\alpha = \eta_{\alpha\beta}\rho^\beta$):

$$-\frac{1}{2i}(\bar{\chi}\psi\delta_{AB} + \bar{\chi}\rho_\alpha\psi\rho^\alpha_{AB} + \bar{\chi}\rho_3\psi\rho_{3AB}) =$$

$$= -\frac{1}{2}((\chi_2\psi_1 - \chi_1\psi_2)\delta_{AB} + (\chi_1\psi_1 + \chi_2\psi_2)\rho^0_{AB} + (\chi_2\psi_2 - \chi_1\psi_1)\rho^1_{AB} + (\chi_2\psi_1 + \chi_1\psi_2)\rho_{3AB}).$$

It should be equal to

$$\frac{1}{i}\psi_A\bar{\chi}_B = \psi_A\rho^0_{CB}\chi_C,$$

and it really does, which may be proved for concrete A, B values.

`Problem 4.9`

Consider general world-sheet supersymmetric RNS action

$$S = -\frac{1}{2\pi}\int d^2\sigma(\partial_\alpha X_\mu\partial^\alpha X^\mu + \bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu),$$

possessing Lorentz symmetry. Bosonic part was already explored in Problem 2.8. Here we are going to study fermionic part. Due to the first Noether theorem space-time Lorentz symmetry transformation of index μ (not world-sheet transformation of index α, for which fermions transform as spinors) with parameters $\omega_{\mu\nu}$ leads to Noether current (this is a transformation of fields X^μ, ψ^μ, for which coordinates of manifold they are defined on - world-sheet - remain unchanged - active transformations)

$$K^\alpha_{\mu\nu} = \frac{\partial L}{\partial(\partial_\alpha\psi^\rho)}\frac{\partial\psi^\rho}{\partial\omega^{\mu\nu}},$$

where for transformation of field $\psi^\rho(x)$ we would have

$$\psi'^\rho(\sigma) = \psi^\rho(\sigma) + \omega_{\mu\nu}\frac{\delta\psi^\rho}{\delta\omega_{\mu\nu}},$$

where we don't transform internal σ-coordinates of world-sheet, which for active transformation variation means

$$\delta\psi^\rho(\sigma) = \omega_{\mu\nu}\frac{\delta\psi^\rho}{\delta\omega_{\mu\nu}}.$$

At the same time for Lorentz transformations

$$\delta\psi^\rho = \omega^{\rho\mu}\psi_\mu.$$

All these facts lead us to conclusion that fermionic Noether current for Lorentz transformation is

$$K^\alpha_{\mu\nu} = \bar{\psi}_\nu\rho^\alpha\psi_\mu.$$

Actually if we complete the action by total derivative to make it explicitly real we will result in

$$K^\alpha_{\mu\nu} = \frac{1}{2}\left(\bar{\psi}_\nu\rho^\alpha\psi_\mu - \bar{\psi}_\mu\rho^\alpha\psi_\nu\right).$$

Noether charge is given by expression

$$K_{\mu\nu} = \int_0^\pi d\sigma K^0_{\mu\nu} = \frac{1}{2}\int_0^\pi d\sigma\left(\bar{\psi}_\nu\rho^0\psi_\mu - \bar{\psi}_\mu\rho^0\psi_\nu\right) = \frac{i}{2}\int_0^\pi d\sigma\left(\psi^T_\nu\psi_\mu - \psi^T_\mu\psi_\nu\right).$$

Now we proceed to open string in NS sector:

$$\psi^\mu = \begin{pmatrix}\psi^\mu_- \\ \psi^\mu_+\end{pmatrix} = \frac{1}{\sqrt{2}}\sum_r\begin{pmatrix}b^\mu_r e^{-ir(\tau-\sigma)} \\ b^\mu_r e^{-ir(\tau+\sigma)}\end{pmatrix}.$$

Substitue this expansion into expression for fermionic Lorentz generator:

$$K^{\mu\nu} = \frac{i}{2}\int_0^\pi d\sigma\sum_{r,p}\left(b^\nu_r b^\mu_p - b^\mu_r b^\nu_p\right)\cos((r+p)\sigma)e^{-i(r+p)\tau} = \frac{i}{2}\sum_r\left(b^\nu_r b^\mu_{-r} - b^\mu_r b^\nu_{-r}\right).$$

This may be rewritten as

$$K^{\mu\nu} = i\sum_{r>0}\left(b^\nu_r b^\mu_{-r} - b^\mu_r b^\nu_{-r}\right).$$

Finally we can write down these generators in light-cone quantization. There we have $b^+_r = 0$. Therefore ($i = 2, \ldots, 9$):

$$K^{ij} = i\sum_{r>0}\left(b^i_r b^j_{-r} - b^j_r b^i_{-r}\right), \quad K^{\pm i} = 0, \quad K^{+-} = 0.$$

Note that using literally the same contemplations as in Problem 2.9 it may be verified commutativity of Lorentz generators with Virasoro operators.

`Problem` **4.10**

In this solution we begin with construction of RNS theory in conformal coordinates.

We focus on open string NS sector. We don't consider R sector because in this problem it's not asked to verify expansion of F_m coefficients through oscillator modes. We can do almost the same for the closed string case quite literally repeating contemplations provided here with just slight appropriate redefinitions.

First of all note that change $\tau \to -i\tau$ (that is we make change of variables: $\tau' = i\tau$ and then from notation τ' go to notation τ, that is make simple replacement $\tau' \to \tau$), Wick rotation and proceeding to conformal coordinates $z = e^{\tau - i\sigma}$ gives:

$$\partial_- = \frac{\partial}{\partial\sigma^-} = \frac{\partial}{\partial(\tau-\sigma)} \to \frac{\partial}{\partial(-i\tau-\sigma)} = i\frac{\partial}{\partial(\tau-i\sigma)} = i\frac{\partial z}{\partial(\tau-i\sigma)}\frac{\partial}{\partial z} = iz\partial_z.$$

Therefore the following open-string expansion (in $l_s = 1$ units):

$$\partial_\pm X^\mu = \frac{1}{2}\sum_m \alpha_m^\mu e^{-im(\tau\pm\sigma)}$$

after change $\tau \to -i\tau$, Wick rotation and proceeding to Riemann plane transforms to

$$iz\partial X^\mu(z) = \frac{1}{2}\sum_m \alpha_m^\mu z^{-m} \to \partial X^\mu(z) = -\frac{i}{2}\sum_m \alpha_m^\mu z^{-m-1}.$$

Therefore we can figure out expansion for bosonic part of energy-momentum tensor coefficients:

$$L_m^{(b)} = \frac{1}{2\pi i}\oint dz z^{m+1}(-2 : \partial X\cdot\partial X :) = \frac{1}{2}\sum_n : \alpha_{-n}\cdot\alpha_{m+n} : .$$

Note that we also have

$$\partial_+ = i\bar{z}\bar{\partial}.$$

Jacobian of transition $(\tau,\sigma) \to (z,\bar{z})$ is equal to $\det\frac{\partial(z,\bar{z})}{\partial(\tau,\sigma)} = 2iz\bar{z}$. In the last formula τ is actually τ', that is we made change $\tau \to -i\tau$ before transition to conformal coordinates. Change $\tau \to -i\tau$ gave (again, we actually perfom change of coordinates $\tau = -i\tau'$ and then rename $\tau \to \tau'$)

$$d\tau d\sigma = \frac{d\tau}{d\tau'}d\tau' d\sigma = -id\tau' d\sigma \to -id\tau d\sigma.$$

Therefore surface area in new and old coordinates are related by formula

$$d\tau d\sigma = -\frac{d^2z}{2z\bar{z}},$$

where τ in l.h.s. of the last equation is original world-sheet time coordinate. This formula is supposed to be substituted into string action.

Original Polyakov bosonic action $S_b = -\frac{1}{2\pi}\int d^2\sigma\partial_\alpha X\cdot\partial^\alpha X$ due to

$$\partial_\alpha X\cdot\partial^\alpha X = 4z\bar{z}\partial X\cdot\bar{\partial}X$$

in conformal coordinates $(z,\bar{z})$ is written as $S_b = \frac{1}{\pi}\int d^2z\partial X\cdot\bar{\partial}X$.

Now consider fermionic action: $S_f = -\frac{1}{2\pi}\int d^2\sigma\bar{\psi}\cdot\rho^\alpha\partial_\alpha\psi$. First of all we observe that

$$\rho^\alpha\partial_\alpha = \begin{pmatrix} 0 & -2iz\partial \\ 2i\bar{z}\partial & 0 \end{pmatrix},$$

where the fact $\partial_\tau = \frac{\partial \tau'}{\partial \tau}\partial_{\tau'} = i\partial_{\tau'} \to i\partial_\tau$ was taken into account. Dirac equations of motion $\partial_+\psi_- = 0$ for original spinors $\psi = \binom{\psi_-}{\psi_+}$ mean that $\psi_- \to \psi(z),\ \psi_+ \to \tilde{\psi}(\bar{z})$. Original action transforms to $S_f = -\frac{i}{2\pi}\int d^2z \left(\frac{1}{z}\psi\bar{\partial}\psi + \frac{1}{\bar{z}}\tilde{\psi}\partial\tilde{\psi}\right)$. The appearence of extra multipliers inside action should be eliminated by a special change $\psi \to \frac{1}{\sqrt{z}}\psi,\ \tilde{\psi} \to \frac{1}{\sqrt{\bar{z}}}\tilde{\psi}$. This will transform $\psi_-^\mu = \frac{1}{\sqrt{2}}\sum_r b_r^\mu e^{-ir(\tau-\sigma)}$ (together with Wick rotation and proceeding to conformal coorinates) to

$$\psi^\mu(z) = \frac{1}{\sqrt{2}}\sum_r b_r^\mu z^{-r-\frac{1}{2}}.$$

This matches nicely with the fact that fermionic field is a primary field with conformal dimension $h = \frac{1}{2}$. Before proceeding to computation of expansion of G_r we also change normalization of fermions to make the action look like

$$S_f = \frac{1}{4\pi}\int d^2z \left(\psi\bar{\partial}\psi + \tilde{\psi}\partial\tilde{\psi}\right).$$

This will lead to a total action of world-sheet supersymmetric open string in conformal coordinates

$$S = \frac{1}{2\pi}\left(2\partial X \cdot \bar{\partial}X + \frac{1}{2}\psi\bar{\partial}\psi + \frac{1}{2}\tilde{\psi}\partial\tilde{\psi}\right). \tag{4.38}$$

Now from Lagrangian (4.38) we can deduce bosonic SET, which is Noether current corresponding to translation of world-sheet coordinates

$$T_B(z) = -2\partial X(z)\cdot\partial X(z) - \frac{1}{2}\psi(z)\cdot\partial\psi(z).$$

At the same time

$$-\frac{1}{2}\psi(z)\cdot\partial\psi(z) = \sum_m \frac{L_m^{(f)}}{z^{m+2}},$$

while bosonic part have been already described above. Using Cauchy's theorem we can obtain mode expansion of $L_m^{(f)}$:

$$L_m^{(f)} = \frac{1}{2}\sum_r \left(r + \frac{m}{2}\right) : b_{-r}\cdot b_{m+r} : .$$

And finally from supercurent mode expansion

$$2i\psi(z)\cdot\partial X(z) = \sum_r \frac{G_r}{z^{r+3/2}}$$

we will obtain

$$G_r = \sum_p b_p \cdot \alpha_{r-p}.$$

Problem 4.11

We will use formulae for fermionic and bosonic SETs:

$$T_F(z) = 2i\psi^\mu(z)\partial X_\mu(z),$$

$$T_B(z) = -2\partial X \cdot \partial X - \frac{1}{2}\psi \cdot \partial\psi,$$

Wick theorem and formulae for bosonic and fermionic fields correlation functions (OPEs). As a result we will obtain the following OPE (dots represent non-singular - normal-ordered - terms):

$$T_F(z)T_F(w) = -4 : \psi^\mu(z)\partial X_\mu(z)\psi^\nu(w)\partial X_\nu(w) : -4\langle\psi^\mu(z)\psi^\nu(w)\rangle : \partial X_\mu(z)\partial X_\nu(w) : -$$

$$-4\langle\partial X_\mu(z)\partial X_\nu(w)\rangle : \psi^\mu(z)\psi^\nu(w) : -4\langle\psi^\mu(z)\psi^\nu(w)\rangle\langle\partial X_\mu(z)\partial X_\nu(w)\rangle =$$

$$= -4\frac{: \partial X(z) \cdot \partial X(w) :}{z-w} + \frac{: \psi(z) \cdot \psi(w) :}{(z-w)^2} + \frac{D}{(z-w)^3} + \cdots =$$

$$= -4\frac{: \partial X(w) \cdot \partial X(w) :}{z-w} - \frac{: \psi(w) \cdot \partial\psi(w) :}{z-w} + \frac{D}{(z-w)^3} + \cdots = \frac{D}{(z-w)^3} + 2\frac{T_B(w)}{z-w} + \cdots .$$

Problem 4.13

We will use field equations actively to reduce immediately a huge amount of terms in δS variation:

$$\bar\partial\psi^\mu = 0, \quad \bar\partial c = \bar\partial\gamma = 0, \quad \bar\partial\partial X^\mu = 0.$$

As a result we will obtain

$$2\pi\delta S = 2\eta \int d^2z \partial\left(c\partial X \cdot \bar\partial X\right).$$

Therefore total action $S = S_{matter} + S_{ghost}$ possesses on-shell BRST symmetry.

Problem 4.14

(i) It's known that a pair of ghost-antighost fields $c,\ b$ with conformal dimensions $1-\lambda,\ \lambda$ respectively possesses energy-momentum tensor (3.28):

$$T_{bc}(z) = -\lambda : b(z)\partial c(z) : +\varepsilon(\lambda - 1) : c(z)\partial b(z) : ,$$

where $\varepsilon = +1$ for fermionic ghosts and $\varepsilon = -1$ for bosonic ghosts. In superconformal theory besides $(b,\ c)$ fermionic ghosts with $(2,1)$ conformal dimensions we also have (β,γ) bosonic ghosts with $\left(\frac{3}{2},\frac{1}{2}\right)$ conformal dimensions. Therefore we got the following formula for total bosonic SET:

$$T_B^{gh} =: -2b\partial c + c\partial b - \frac{3}{2}\beta\partial\gamma - \frac{1}{2}\gamma\partial\beta : .$$

For bosonic ghost primary fields we define expansions corresponding to values of their conformal weights:

$$\beta(z) = \sum_n \frac{\beta_n}{z^{n+3/2}}, \qquad \gamma(z) = \sum_n \frac{\gamma_n}{z^{n-1/2}}.$$

Using these formulae and Cauchy's theorem we can determine modes of expansion $T_{\beta\gamma} = \sum_n \frac{L^{gh}_{k(\beta\gamma)}}{z^{k+2}}$:

$$L^{gh}_{k(\beta\gamma)} = \sum_m \left(\frac{k}{2} - m\right)\beta_{k+m}\gamma_{-m}.$$

In a similar way it may be found a fermionic ghost contribution

$$L^{gh}_{m(bc)} = \sum_n (m-n) b_{m+n} c_{-n}.$$

Total ghost mode contribution to energy-momentum tensor is therefore

$$L^{gh}_m = \sum_n (m-n) : b_{m+n} c_{-n} : + \sum_n \left(\frac{m}{2} - n\right) : \beta_{m+n} \gamma_{-n} : .$$

Now let's use holomorphic fermionic energy-momentum tensor (Noether supercurrent, corresponding to supertranslations)

$$T^{gh}_F = -2b\gamma + c\partial\beta + \frac{3}{2}\beta\partial c$$

and Cauchy's theorem to derive mode components of expansion $T^{gh}_F = \sum_r \frac{G_r}{z^{r+3/2}}$. Ghost contribution may be easily calculated with the help of known Laurent expansions of primary fields $b,\ c,\ \beta,\ \gamma$. It is given by equation

$$G^{gh}_r = -\sum_n \left(2b_n\gamma_{r-n} + \left(n + \frac{3}{2}\right) c_{r-n}\beta_n + \frac{3}{2}(n-1)\beta_{r-n}c_n\right) =$$

$$= -2\sum_n \left(b_n\gamma_{r-n} + \left(r - \frac{n}{2}\right)\beta_{r-n}c_n\right).$$

We have used commutativity of β_r and c_n modes.
(ii) In the Problem 3.11 it was derived general formula

$$c(\varepsilon,\ \lambda) = -2\varepsilon(6\lambda^2 - 6\lambda + 1).$$

for ghost energy-momentum tensor (3.28). We can use it to calculate contribution of bosonic ghosts to central charge: it's equal to 11. Contribution of fermionic ghosts is known to be equal to -26, and contribution of bosonic+fermionic world sheet fields is $\frac{3D}{2}$. Requiring total central charge to be zero we will obtain the condition on space-time dimension: $D = 10$.

Problem 4.15

We will deal with NS sector but the solution is literally the same in the case of R sector: just replace all half-integer $r-$ type indices by integer $m-$ type ones and G by F.

BRST charge has a form of

$$Q_B = \frac{1}{2\pi i}\oint dz \left(cT^{matter}_B + \gamma T^{matter}_F + bc\partial c - \frac{1}{2}c\gamma\partial\beta - \frac{3}{2}c\beta\partial\gamma - b\gamma^2\right).$$

It has ghost number $+1$ (the number of c-ghosts minus the number of b-antighosts) and conformal ghost number $+1$ (number of ghosts $\gamma - \beta$). For general energy-momentum tensor and primary field $\Phi(z)$ with conformal weight h it was derived in the solution of Problem 3.7 the following relation:

$$[L_m, \Phi_n] = (m(h-1) - n)\Phi_{m+n}.$$

Using this we can obtain

$$[L_m, b_n] = (m-n)b_{m+n}, \qquad [L_m, \beta_r] = \left(\frac{m}{2} - r\right)\beta_{m+n}.$$

From BRST transformations it directly follows that

$$\{Q_B, b(z)\} = T_B(z), \qquad [Q_B, \beta(z)] = T_F(z),$$

from which follows

$$\{Q_B, b_n\} = L_n - a_{NS}\delta_{n,0}, \qquad [Q_B, \beta_r] = G_r.$$

Now we have a total anomaly-free superextended Virasoro algebra

$$[L_m, L_n] = (m-n)(L_{m+n} - a_{NS}\delta_{m+n,0}), \qquad [L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r}.$$

With the help of all pointed out above we can prove that r.h.s. of the following equations vanish:

$$\{[Q_B, L_m], b_n\} = \{[L_m, b_n], Q_B\} + [\{b_n, Q_B\}, L_m] = 0,$$
$$\{[Q_B, L_m], \beta_r\} = \{[L_m, \beta_r], Q_B\} + [\{\beta_r, Q_B\}, L_m] = 0.$$

Therefore $[Q_B, L_m]$ being non-zero can not contain c and γ ghosts, but it's known to have a positive ghost and conformal ghost numbers. Thus $[Q_B, L_m] = 0$. Treating of $\{[Q_B, G_r], b_n\}$ and $\{[Q_B, G_r], \beta_r\}$ in a similar manner proves them to be zero too. It follows that

$$[Q_B^2, b_n] = \frac{1}{2}[\{Q_B, Q_B\}, b_n] = \frac{1}{2}[\{Q_B, b_n\}, Q_B] + \frac{1}{2}[\{b_n, Q_B\}, Q_B] = [L_n, Q_B] = 0, \tag{4.39}$$

$$[Q_B^2, \beta_n] = \frac{1}{2}[\{Q_B, Q_B\}, \beta_n] = \frac{1}{2}\{[Q_B, \beta_n], Q_B\} + \frac{1}{2}[\{\beta_n, Q_B\}, Q_B] = [G_r, Q_B] = 0, \tag{4.40}$$

where we've used graded Jacobi identities. The last two expressions lead to the conclusion, that Q_B^2 being non-zero can not contain c and γ ghosts. But it has $+1$ ghost number and $+1$ conformal ghost number. Therefore $Q_B^2 = 0$.

5 Strings with space-time supersymmetry

Problem 5.1

Supersymmetric particle action is given by

$$S = S_1 + S_2 = -m\int d\tau\sqrt{-\Pi_0\cdot\Pi_0} - m\int d\tau\bar{\Theta}\Gamma_{11}\dot{\Theta}.$$

Momentum of particle is proportional to

$$\Pi_0^\mu = \dot{X}^\mu - \bar{\Theta}^A\Gamma^\mu\dot{\Theta}^A = \left(\frac{\partial X'^\mu}{\partial\tau'} - \bar{\Theta}'^A\Gamma^\mu\frac{\partial\Theta'^A}{\partial\tau'}\right)\frac{\partial\tau'}{\partial\tau},$$

where

$$X'^\mu(\tau') = X^\mu(\tau), \qquad \Theta'^A(\tau') = \Theta^A(\tau).$$

Therefore

$$S_1 \to -m \int d\tau' \frac{\partial \tau}{\partial \tau'} \sqrt{-\Pi_0' \cdot \Pi_0' \left(\frac{\partial \tau'}{\partial \tau}\right)^2} = S_1, \qquad S_2 \to -m \int d\tau' \frac{\partial \tau}{\partial \tau'} \bar{\Theta}' \Gamma_{11} \frac{\partial \Theta'}{\partial \tau'} \frac{\partial \tau'}{\partial \tau} = S_2,$$

which ends the proof of reparametrization invariance.

Problem 5.2

(i) Massive point particle may be described by action

$$S_0 = \frac{1}{2} \int d\tau \left(\frac{1}{e} \dot{X}^2 - m^2 e \right).$$

From this we can get action for supersymmetric action by replacing $\dot{X}^\mu \to \dot{X}^\mu - \bar{\Theta}^A \Gamma^\mu \dot{\Theta}^A$:

$$S = \frac{1}{2} \int d\tau \left(\frac{1}{e} (\dot{X}^\mu - \bar{\Theta}^A \Gamma^\mu \dot{\Theta}^A)^2 - m^2 e \right).$$

(ii) Massless limit gives an action

$$S = \frac{1}{2} \int d\tau \frac{1}{e} (\dot{X}^\mu - \bar{\Theta}^A \Gamma^\mu \dot{\Theta}^A)^2 = \frac{1}{2} \int d\tau \frac{1}{e} \Pi_0^2,$$

equation of motion for auxiliary field $e(\tau)$ gives constraint $\Pi_0^2 = 0$.

(iii) In the theory with action possessing an auxiliary field the view of local κ-symmetry may be represented as:

$$\delta\Theta^A = \Gamma \cdot \Pi_0 \kappa^A, \quad \delta X^\mu = \bar{\Theta}^A \Gamma^\mu \delta \Theta^A, \quad \delta e = 4e \bar{\kappa}^A \dot{\Theta}^A. \tag{5.41}$$

Because of $(\Gamma \cdot p)^2 = 0$ means half-degeneracy of matrix $\Gamma \cdot p$ half of Θ^A components may be fixed. We can notice that (5.41) transformations for point particle give $\delta \Pi_0^\mu = 2\bar{\kappa}^A \Gamma \cdot \Pi_0 \Gamma^\mu \dot{\Theta}^A$. This leads to $\delta S = 0$ even without implicit use of $\Pi_0^2 = 0$, i.e. off-shell. This simple fact may be verified by the following line of calculations:

$$\delta \Pi_0^2 = 2\Pi_{0\mu} \delta \Pi_0^\mu = 4\Pi_{0\mu} \bar{\kappa}^A \Gamma \cdot \Pi_0 \Gamma^\mu \dot{\Theta}^A = 4\bar{\kappa}^A (\Gamma \cdot \Pi_0)^2 \dot{\Theta}^A = 2\bar{\kappa}^A (\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu) \Pi_{0\mu} \Pi_{0\nu} \dot{\Theta}^A =$$
$$= 4\Pi_0^2 \bar{\kappa}^A \dot{\Theta}^A.$$

Or we can consider alternative representation of κ-transformation of spinor

$$\delta \Theta^A = \bar{\kappa}^A P_-$$

and corresponding κ-transformation of auxiliary field $e(\tau)$ to make action invariant:

$$\delta e = -\frac{4e}{\Pi_0^2} \bar{\kappa}^A P_- \Gamma \cdot \Pi_0 \dot{\Theta}^A.$$

Bosonic coordinate transformation is the same as in (5.41) (with appropriate $\delta\Theta^A$). In this case κ transformation of spinor Θ differs from that considered above by supersymmetry translation. This follows from the definition of P_-:

$$P_- = \frac{1}{2}\left(1 - \frac{\Gamma \cdot \Pi_0}{\sqrt{-\Pi_0^2}} \Gamma_{11}\right).$$

Therefore our calculations above automatically prove invariance of the action under κ-transformations written in the form BBS (5.32).

Problem 5.3
We know, that for $C = \Gamma^0$ we got $C\Gamma_\mu = -\Gamma_\mu^T C$ in Majorana representation ($\Gamma_\mu^\dagger = \Gamma_\mu^T$). Therefore

$$C\Gamma_{\mu_1\cdots\mu_n}C^{-1} = C\Gamma_{[\mu_1}\Gamma_{\mu_2}\cdots\Gamma_{\mu_n]}C^{-1} = C\Gamma_{[\mu_1}C^{-1}C\Gamma_{\mu_2}C^{-1}\cdots C\Gamma_{\mu_n]}C^{-1} =$$
$$= (-1)^n\Gamma^T_{[\mu_1}\Gamma^T_{\mu_2}\cdots\Gamma^T_{\mu_n]} = (-1)^n\Gamma^T_{\mu_n\cdots\mu_1},$$

therefore

$$C\Gamma_{\mu_1\cdots\mu_n} = (-1)^n\Gamma^T_{\mu_n\cdots\mu_1}C.$$

We will use it bellow (remember also that any time we permute spinors we have to insert minus sign and that $C^T = -C$):

$$\bar{\Theta}_1\Gamma_{\mu_1\cdots\mu_n}\Theta_2 = \bar{\Theta}_1\Gamma_{[\mu_1}\cdots\Gamma_{\mu_n]}\Theta_2 = \Theta_1^T C\Gamma_{[\mu_1}\cdots\Gamma_{\mu_n]}\Theta_2 =$$
$$= -\Theta_2^T\Gamma^T_{[\mu_n}\cdots\Gamma^T_{\mu_1]}C^T\Theta_1 = \Theta_2^T\Gamma^T_{[\mu_n}\cdots\Gamma^T_{\mu_1]}C\Theta_1 =$$
$$= \Theta_2^T\Gamma^T_{\mu_1\cdots\mu_n}C\Theta_1 = (-1)^n\bar{\Theta}_2\Gamma_{\mu_n\cdots\mu_1}\Theta_1 =$$
$$= (-1)^{n+\frac{n(n-1)}{2}}\bar{\Theta}_2\Gamma_{\mu_1\cdots\mu_n}\Theta_1 = (-1)^{\frac{n(n+1)}{2}}\bar{\Theta}_2\Gamma_{\mu_1\cdots\mu_n}\Theta_1.$$

In the case of $n = 1$ we get $(-1)^{\frac{n(n+1)}{2}} = -1$ and return to the formula

$$\bar{\Theta}_1\Gamma_\mu\Theta_2 = -\bar{\Theta}_2\Gamma_\mu\Theta_1.$$

Problem 5.4
In the proof of SUSY invariance of action S_2 (Problem 5.7) it's pointed out the following identity:

$$\Gamma^\mu\psi_{[1}\bar{\psi}_2\Gamma_\mu\psi_{3]} = 0,$$

where it's assumed antisymmetrization over spinors. Because of

$$\Gamma^\mu d\Theta d\bar{\Theta}\Gamma_\mu d\Theta = \Gamma^\mu\Theta_{,\lambda}\bar{\Theta}_{,\rho}\Gamma_\mu\Theta_{,\sigma}dx^\lambda \wedge dx^\rho \wedge dx^\sigma$$

assumes antisymmetrization too (for set of different values of $\lambda,\ \rho,\ \sigma$), we conclude, that this value is zero.

Problem 5.5
There're at least two ways to prove invariance of the action. First is to use expression

$$S_2 = \int_M \Omega_2 = \int_D \Omega_3,$$

where M is world-sheet, which after diffeomorphism transformation and possibly (in the case of open string) glueing of edges, becomes the border of some D. According to Stokes theorem $\Omega_3 = d\Omega_2$, and we know Ω_2 because we know action. Therefore (see BBS (5.44))

$$\Omega_3 = c(d\bar{\Theta}^1\Gamma_\mu d\Theta^1 - d\bar{\Theta}^2\Gamma_\mu d\Theta^2)\Pi^\mu.$$

This is SUSY-invariant 3-form, because it's composed of invariant forms $d\Theta^A,\ \Pi^\mu$. Therefore $\delta S = 0$. Let's prove used here formula BBS (5.44). Observe that when we take $d\Omega_2$ basing on BBS (5.55) expression we can add and subtract term $\bar\Theta^A\Gamma^\mu d\Theta^A$ to accomplish dX^μ to Π^μ 1-form. Then we will get

$$\begin{gathered}
d\Omega_2 = (d\bar\Theta^1\Gamma_\mu d\Theta^1 - d\bar\Theta^2\Gamma_\mu d\Theta^2)(dX^\mu - \bar\Theta^A\Gamma^\mu d\Theta^A) + \\
+(d\bar\Theta^1\Gamma_\mu d\Theta^1 - d\bar\Theta^2\Gamma_\mu d\Theta^2)(\bar\Theta^1\Gamma^\mu d\Theta^1 + \bar\Theta^2\Gamma^\mu d\Theta^2) - \\
-d\bar\Theta^1\Gamma_\mu d\Theta^1\bar\Theta^2\Gamma^\mu d\Theta^2 - \bar\Theta^1\Gamma_\mu d\Theta^1 d\bar\Theta^2\Gamma^\mu d\Theta^2 = \\
= \Omega_3 - \bar\Theta^1\Gamma^\mu d\Theta^1 d\bar\Theta^1\Gamma_\mu d\Theta^1 + \bar\Theta^2\Gamma^\mu d\Theta^2 d\bar\Theta^2\Gamma_\mu d\Theta^2.
\end{gathered}$$

In these calculations we have used the fact that all products of 1-forms are wedges.

Second way assumes direct calculation: substitution of SUSY transformations

$$\delta\Theta^A = \varepsilon^A, \qquad \delta X^\mu = \bar\varepsilon^A\Gamma^\mu\Theta^A$$

inside variation of action. All such variations use property of antysymmetrization - wedge product - which makes fermions permutable with no change of sign. How this may be done (on an example of SUSY invariance) is studied in the solution of Problem 5.7.

`Problem` **5.6**

The simplest way to solve this problem is to notice that while in $\Gamma_{\mu_1\nu_1\mu_2\nu_2}$ we have antisymmetrization over all indices, in $\{\Gamma_{\mu_1\nu_1},\ \Gamma_{\mu_2\nu_2}\}$ antisymmetrization is performed only among first and second pairs of indices. Therefore in the relation between these two values should be presented anticommutators of gamma-matrices with extra 'antisymmetrizations', which is actually symmetrization in each concrete term because we only have $\eta_{\mu\nu}$-value to employ for any generic equation. Therefore we got

$$\Gamma_{\mu_1\nu_1\mu_2\nu_2} = a\{\Gamma_{\mu_1\nu_1},\ \Gamma_{\mu_2\nu_2}\} + b\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2} + c\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2}.$$

For $\mu_1 = \mu_2 = \nu_1 = \nu_2$ all gamma-matrices perish, and for equation to hold we ought to impose $c = -b$. For $\mu_1 = 0,\ \nu_1 = 1,\ \mu_2 = 2,\ \nu_2 = 3$ we have $\{\Gamma_{\mu_1\nu_1},\ \Gamma_{\mu_2\nu_2}\} = 2\Gamma_{0123}$ (because antisymmetrization assumes division by factorial) and $\Gamma_{\mu_1\nu_1\mu_2\nu_2} = \Gamma_{0123}$. Due to $c = -b$ eta-terms cancel each other, and we are to impose $a = 2$. Therefore we have obtained

$$\Gamma_{\mu_1\nu_1\mu_2\nu_2} = 2\{\Gamma_{\mu_1\nu_1},\ \Gamma_{\mu_2\nu_2}\} + b\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2} - b\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2}. \tag{5.42}$$

Consider now $\mu_1 = \mu_2 = 0,\ \nu_1 = \nu_2 = 1$. We have $\{\Gamma_{\mu_1\nu_1},\ \Gamma_{\mu_2\nu_2}\} = 2\Gamma_0\Gamma_1\Gamma_0\Gamma_1 = -2\Gamma_0\Gamma_0\Gamma_1\Gamma_1 = 2I$ and r.h.s. of (5.42) is equal to $-b$. Therefore $b = -2,\ c = 2$ and

$$\Gamma_{\mu_1\nu_1\mu_2\nu_2} = 2\{\Gamma_{\mu_1\nu_1},\ \Gamma_{\mu_2\nu_2}\} - 2\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2} + 2\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2}.$$

`Problem` **5.7**

We already sure that the action S_2 is supersymmetric, because

$$S_2 = \int_M \Omega_2 = \int_D \Omega_3,$$

where M is world-sheet and Ω_3 is composed of SUSY-invariant 1-forms. But we can also show supersymmetry invariance explicitly. We have the action

$$S_2 = \frac{1}{\pi}\int d^2\sigma\varepsilon^{\alpha\beta}[-\partial_\alpha X^\mu(\bar{\Theta}^1\Gamma_\mu\partial_\beta\Theta^1 - \bar{\Theta}^2\Gamma_\mu\partial_\beta\Theta_2) - \bar{\Theta}^1\Gamma^\mu\partial_\alpha\Theta^1\bar{\Theta}^2\Gamma_\mu\partial_\beta\Theta^2]$$

which may be rewritten in a geometrical form

$$S_2 = \int_M \Omega_2 = \frac{1}{\pi}\int_M \left((\bar{\Theta}^1\Gamma_\mu d\Theta^1 - \bar{\Theta}^2\Gamma_\mu d\Theta^2)dX^\mu - \bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\Theta}^2\Gamma^\mu d\Theta^2\right).$$

We assume wedge products everywhere where differentials (1-forms) are multiplied. Otherwise we would have $d\bar{\Theta}^1\Gamma_\mu d\Theta^1 = -d\bar{\Theta}^1\Gamma_\mu d\Theta^1 = 0$ due to expression for Majorana spinors $\bar{\chi}\Gamma\psi = -\bar{\psi}\Gamma\chi$ (see solution to Ex. 5.1, which is generalized in the solution of Problem 5.3 here). But with wedge product $\chi\wedge\psi = \psi\wedge\chi$.

Substitution of supersymmetry transformations

$$\delta\Theta^A = \varepsilon^A, \qquad \delta X^\mu = \bar{\varepsilon}^A\Gamma^\mu\Theta^A$$

inside the variation of action S_2 results in

$$\begin{aligned}
\pi\delta\Omega_2 &= d(\bar{\varepsilon}^1\Gamma_\mu d\Theta^1 X^\mu - \bar{\varepsilon}^2\Gamma_\mu d\Theta^2 X^\mu) + \\
&+\bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\varepsilon}^1\Gamma^\mu d\Theta^1 + \bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\varepsilon}^2\Gamma^\mu d\Theta^2 - \bar{\Theta}^2\Gamma_\mu d\Theta^2\bar{\varepsilon}^1\Gamma^\mu d\Theta^1 - \bar{\Theta}^2\Gamma_\mu d\Theta^2\bar{\varepsilon}^2\Gamma^\mu d\Theta^2 - \\
&-\bar{\varepsilon}^1\Gamma^\mu d\Theta^1\bar{\Theta}^2\Gamma_\mu d\Theta^2 - \bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\varepsilon}^2\Gamma^\mu d\Theta^2 = \\
&= d(\bar{\varepsilon}^1\Gamma_\mu d\Theta^1 X^\mu - \bar{\varepsilon}^2\Gamma_\mu d\Theta^2 X^\mu) + \\
&+\bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\varepsilon}^1\Gamma^\mu d\Theta^1 - \bar{\Theta}^2\Gamma_\mu d\Theta^2\bar{\varepsilon}^1\Gamma^\mu d\Theta^1 - \bar{\Theta}^2\Gamma_\mu d\Theta^2\bar{\varepsilon}^2\Gamma^\mu d\Theta^2 - \bar{\varepsilon}^1\Gamma^\mu d\Theta^1\bar{\Theta}^2\Gamma_\mu d\Theta^2.
\end{aligned}$$

Because of wedge products are assumed here last and pre-pre-last terms cancel each other: $\bar{\Theta}^2\Gamma_\mu d\Theta^2$ and $\bar{\varepsilon}^1\Gamma^\mu d\Theta^1$ are bosons and they are permutable therefore, but they are 1-forms at the same time, therefore they ought to be antisymmetrized, if multiplied by wedge product; but they are symmetrized.

Therefore to show superinvariance of the action we must explore terms of the type

$$A = \bar{\varepsilon}\Gamma^\mu d\Theta\bar{\Theta}\Gamma_\mu d\Theta.$$

This may be rewritten as

$$A = (A_1 + A_2)d^2\sigma,$$

where

$$A_1 = \frac{2}{3}\left(\bar{\varepsilon}\Gamma^\mu\dot{\Theta}\bar{\Theta}\Gamma_\mu\Theta' + \bar{\varepsilon}\Gamma^\mu\Theta'\dot{\bar{\Theta}}\Gamma_\mu\Theta + \bar{\varepsilon}\Gamma^\mu\Theta\bar{\Theta}'\Gamma_\mu\dot{\Theta}\right), \tag{5.43}$$

$$\begin{aligned}
A_2 &= \frac{1}{3}\left(\bar{\varepsilon}\Gamma^\mu\dot{\Theta}\bar{\Theta}\Gamma_\mu\Theta' + \bar{\varepsilon}\Gamma^\mu\Theta'\dot{\bar{\Theta}}\Gamma_\mu\Theta - 2\bar{\varepsilon}\Gamma^\mu\Theta\bar{\Theta}'\Gamma_\mu\dot{\Theta}\right) = \\
&= \frac{1}{3}\frac{\partial}{\partial\tau}(\bar{\varepsilon}\Gamma^\mu\Theta\bar{\Theta}\Gamma_\mu\Theta') - \frac{1}{3}\frac{\partial}{\partial\sigma}(\bar{\varepsilon}\Gamma^\mu\Theta\bar{\Theta}\Gamma_\mu\dot{\Theta}).
\end{aligned}$$

Notice that A_2 is a total derivative, and A_1 vanishes because it's of the type

$$\bar{\varepsilon}\Gamma_\mu\psi_{[1}\bar{\psi}_2\Gamma^\mu\psi_{3]}$$

with $\psi = (\Theta,\ \Theta',\ \dot{\Theta})$ and antisymmetrization, assumed by square brackets. This expression is proved to vanish in super Yang-Mills theory (see, e.g., GSW vol.1 App. 4A).

Therefore supersymmetric variation of S_2 is an exact form, and invariance of action depends on boundary conditions on ∂M. But in terms of D-integral action $\int_D \Omega_3$ has invariance independent of any boundary conditions. The thing is that we can retrieve Ω_2 from Ω_3 only up to an exact form: $\Omega_2 \to \Omega_2 + d\omega$, because the only relation between Ω_2 and Ω_3 is $\Omega_3 = d\Omega_2$. Therefore to the variation of Ω_2 it may be added any exact form of the type $d\delta\omega$, which will enable supersymmetry of action $\int_M \Omega_2$ independently of any boundary conditions on ∂M.

`Problem` **5.8**

The superstring action in light cone gauge has the same form as RNS superstring action:

$$S = -\frac{1}{2\pi}\int d^2\sigma(\partial_\alpha X_i \partial^\alpha X^i + \bar{S}^a \rho^\alpha \partial_\alpha S^a), \tag{5.44}$$

where $S^a = (S_1^a,\ S_2^a)$ is 16-component space-time spinor (composed of two Majorana-Weyl spinors with reduced by half number of components due to light-cone gauge) where S_1^a and S_2^a are transformed (in spinor representation) under $Spin(8)$ with two opposite chiralities in type-IIA theory and with the same chiralities in type-IIB theory. In considered here case of type-I superstring $S_1 = S_2$.

After fixing of light-cone gauge we are left with 8-component spinor $S = S_1 = S_2$ but actually in open string we have 16 supersymmetries - the number of Majorana-Weyl spinor components in ten space-time dimensions. It means that we can perform not only supersymmetry transformations which preserve light-cone gauge but also supersymmetry transformations that violate it. The later are to be accompanied with corresponding κ-transformation - according to the very idea of application of κ-symmetry.

Consider first such space-time SUSY transformations, that doesn't violate light-cone gauge of S^a, i.e. such $\delta S^a = \varepsilon^a$ that $\Gamma^+\varepsilon = 0$ (a is spinor index, which is also one of indices of $Spin(8)$ Γ-matrix). This expression eliminates half of the components of 16D ε^a. At the same time from $\Gamma^+\Theta = 0$ for some Θ (being S or ε here) follows $\bar{\Theta}\Gamma^+ = 0$, therefore with the help of inserting $1 = -\frac{1}{2}(\Gamma^+\Gamma^- + \Gamma^-\Gamma^+)$ between $\bar{\varepsilon}$ and Γ^i in the expression $\bar{\varepsilon}\Gamma^i S$ it may be easily shown that $\bar{\varepsilon}\Gamma^i S = 0$. Therefore we have 8-parametric SUSY transformations preserving light-cone gauge:

$$\delta S^a = \varepsilon^a, \qquad \delta X^i = \bar{\varepsilon}\Gamma^i S = 0.$$

Using this transformation laws we can construct Noether current with the help of Noether theorem:

$$\delta S = -\frac{1}{2\pi}\int d^2\sigma \bar{\varepsilon}^a \rho^\alpha \partial_\alpha S^a = \int d^2\sigma \bar{\varepsilon}^a \partial_\alpha J^{a\alpha},$$

from which we can conclude that conserved Noether charge is given by $Q_a = -2\pi \int_0^\pi d\sigma J_a^0 = \int_0^\pi d\sigma \rho^0 S_a$. Zero modes of Noether currents are symmetry algebra generators (this ideology is described on pages 68-69 BBS). Here they are

$$Q^a = \sqrt{2p^+} S_0^a,$$

and they indeed generate appropriate transformations:

$$\delta S^a = \sqrt{2p^+}\varepsilon^a, \qquad \delta X^i = 0.$$

Consider then SUSY transformations that violate light-cone gauge of S^a. Such SUSY transformations should be accompanied with local κ-transformation, which we write in a form similar to that used for point particle in the solution of Problem 5.2. (see GSW (5.134), (5.135)):

$$\delta S = \varepsilon + 2\Gamma \cdot \Pi_\alpha \kappa^\alpha$$

in a way to make total transformation preserving light-cone gauge. If light-cone gauge is preserved it means that space-time superstring action preserves its form (5.44), which is that for world-sheet action (transition to world-sheet makes a to be vector index of $SO(8)$). For the later supersymmetry transformations are local $D = 2$ supersymmetries, given by BBS (4.11), (4.12) (we introduce Γ-matrices here to switch chiralities in an appropriate way: we steal deal with S^a but perform transformations with opposite $Spin(8)$ chirality $\bar{\varepsilon}^{\dot{a}}$). Therefore they are SUSY transformations of the world-sheet theory:

$$\delta S^a = \rho^\alpha \partial_\alpha X^i \Gamma^i_{a\dot{b}} \bar{\varepsilon}^{\dot{b}} \sqrt{p^+}, \qquad \delta X^i = \Gamma^i_{a\dot{b}} \bar{\varepsilon}^{\dot{b}} S^a / \sqrt{p^+}. \tag{5.45}$$

Here Γ-matrices play the role of Clebsh-Gordon coefficients, connecting three representations of $Spin(8)$ for the aim described above:

$$|\dot{a}\rangle = \Gamma^i_{\dot{a}b} S^b_0 |i\rangle, \qquad |i\rangle = \Gamma^i_{\dot{a}b} S^b_0 |\dot{a}\rangle.$$

Generators of (5.45) transformations which are zero-modes of corresponding Noether currents are

$$Q^{\dot{a}} = \frac{1}{\sqrt{p^+}} \Gamma^i_{\dot{a}b} \sum_n S^b_{-n} \alpha^i_n.$$

Supersymmetry generators satisfy anticommutation relations:

$$\{Q^a, Q^b\} = 2p^+ \delta^{ab}, \qquad \{Q^a, Q^{\dot{a}}\} = \sqrt{2} \Gamma^i_{a\dot{a}} p^i, \qquad \{Q^{\dot{a}}, Q^{\dot{b}}\} = 2H \delta^{\dot{a}\dot{b}},$$

where

$$H = \frac{1}{2p^+}((p^i)^2 + 2N),$$

excitation number operator is

$$N = \sum_{m=1}^{\infty} (\alpha^i_{-m} \alpha^i_m + m S^a_{-m} S^a_m).$$

The last formula is a consequence of corresponding formula in the R sector of RNS string.

`Problem` **5.9**

(i) Gauge transformation of strength tensor is given by $\delta_\Lambda F = [F, \Lambda]$, which leads to

$$\delta_\Lambda \mathrm{tr} F \wedge F = \mathrm{tr} \delta_\Lambda F \wedge F + \mathrm{tr} F \wedge \delta_\Lambda F = \mathrm{tr}[F, \Lambda] \wedge F + \mathrm{tr} F \wedge [F, \Lambda] =$$

$$= \Lambda^b (F^a \wedge F^c) \mathrm{tr}([T^a, T^b] T^c) + \Lambda^c (F^a \wedge F^b) \mathrm{tr}(T^a [T^b, T^c]) = 0,$$

as may be shown by using possibility of cyclic permutation inside trace.

Then note that due to definition

$$F = dA + A \wedge A$$

it takes place Bianchi identity

$$dF - F \wedge A + A \wedge F = 0\,.$$

We can use it to express dF in the following line of calculations:

$$d\mathrm{tr}(F \wedge F) = \mathrm{tr}(dF \wedge F + F \wedge dF) =$$

$$= \mathrm{tr}(F \wedge A \wedge F - A \wedge F \wedge F + F \wedge F \wedge A - F \wedge A \wedge F) =$$

$$= F^a \wedge F^b \wedge F^c \mathrm{tr}([\lambda^a \lambda^b, \lambda^c]) = \frac{1}{2} d^{bac} F^a \wedge F^b \wedge A^c = 0\,.$$

The conclusion of equality to zero was drawn from the fact that $F^a \wedge F^b$ is symmetric with respect to a and b indices permutation, while d^{bac} is antisymmetric.
(ii) First of all note, that $\mathrm{tr} A \wedge A \wedge A \wedge A = 0$. Indeed:

$$\mathrm{tr}(A \wedge A \wedge A \wedge A) = A^a_\mu A^b_\nu A^c_\lambda A^d_\rho \mathrm{tr}(\lambda^a \lambda^b \lambda^c \lambda^d) dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho =$$

$$= -A^a_\mu A^b_\nu A^c_\lambda A^d_\rho \mathrm{tr}(\lambda^d \lambda^a \lambda^b \lambda^c) dx^\rho \wedge dx^\mu \wedge dx^\nu \wedge dx^\lambda =$$

$$= -A^b_\nu A^c_\lambda A^d_\rho A^a_\mu \mathrm{tr}(\lambda^a \lambda^b \lambda^c \lambda^d) dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho,$$

where in the last transition we've replaced $(\mu\nu\lambda\rho) \to (\nu\lambda\rho\mu)$ and $(abcd) \to (bcda)$. The real value which is equal to negative of itself is zero.

Therefore

$$\mathrm{tr} F \wedge F = \mathrm{tr} dA \wedge dA + \mathrm{tr}(dA \wedge A \wedge A + A \wedge A \wedge dA). \tag{5.46}$$

From another side

$$d\omega_3 = \mathrm{tr} dA \wedge dA + \mathrm{tr}\left(\frac{2}{3} dA \wedge A \wedge A - \frac{2}{3} A \wedge dA \wedge A + \frac{2}{3} A \wedge A \wedge dA\right). \tag{5.47}$$

One can easily calculate

$$\mathrm{tr} dA \wedge A \wedge A = \frac{1}{2}(\partial_\lambda A^a_\mu - \partial_\mu A^a_\lambda) A^b_\nu A^c_\rho \mathrm{tr}(\lambda^a \lambda^b \lambda^c) dx^\lambda \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho,$$

$$\mathrm{tr} A \wedge dA \wedge A = \frac{1}{2}(\partial_\lambda A^a_\mu - \partial_\mu A^a_\lambda) A^b_\nu A^c_\rho \mathrm{tr}(\lambda^c \lambda^a \lambda^b) dx^\rho \wedge dx^\lambda \wedge dx^\mu \wedge dx^\nu =$$

$$= -\frac{1}{2}(\partial_\lambda A^a_\mu - \partial_\mu A^a_\lambda) A^b_\nu A^c_\rho \mathrm{tr}(\lambda^a \lambda^b \lambda^c) dx^\lambda \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho,$$

$$\mathrm{tr} A \wedge A \wedge dA = \frac{1}{2}(\partial_\lambda A^a_\mu - \partial_\mu A^a_\lambda) A^b_\nu A^c_\rho \mathrm{tr}(\lambda^b \lambda^c \lambda^a) dx^\nu \wedge dx^\rho \wedge dx^\lambda \wedge dx^\mu =$$

$$= \frac{1}{2}(\partial_\lambda A^a_\mu - \partial_\mu A^a_\lambda) A^b_\nu A^c_\rho \mathrm{tr}(\lambda^a \lambda^b \lambda^c) dx^\lambda \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho.$$

Using these formulae we can divide middle term in the second trace of r.h.s. of (5.47) by two and combine each of the resulting sub-terms with first and second terms in that trace. The

result is immediately (5.46).
(iii) There's a general formula for representation of $\mathrm{tr} F \wedge \cdots \wedge F$ with $p+1$ multipliers (being $2(p+1)$-form) as $d\omega_{2p+1}$, where Chern-Simons $2p+1$-form is (see GSW (13.3.35)):

$$\omega_{2p+1} = (p+1) \int_0^1 ds s^p \mathrm{tr} \left(A \left(dA + sA^2 \right)^p \right) .$$

While opening brackets do not permutate multipliers. All products of forms are wedge.

Problem **5.11**
(i) From BBS (5.112) it follows the following expression up to terms of fourth order in curvature:

$$\hat{A}(R) = 1 + \frac{1}{48} \mathrm{tr} R^2 + \dots .$$

From this formula using BBS (5.116) expression

$$\hat{A}(R/2) = \sqrt{L(R/4) \hat{A}(R)}$$

we obtain

$$L(R) = 1 - \frac{1}{6} \mathrm{tr} R^2 + \dots .$$

Because of trace is performed over $N \times N$ matrices of fundamental representation of $SO(N)$, then

$$\mathrm{tr} \cos F = N - \frac{1}{2} \mathrm{tr} F^2 .$$

Using all collected formulae we can obtain the first two order terms of expansion of

$$Y = \frac{1}{2} \sqrt{\hat{A}(R)} \mathrm{tr} \cos F - 16 \sqrt{L(R/4)} = \frac{N-32}{2} - \frac{1}{4} \mathrm{tr} F^2 + \frac{N+16}{192} \mathrm{tr} R^2 ,$$

from which it follows, that

$$Y_4 = \frac{N+16}{192} \mathrm{tr} R^2 - \frac{1}{4} \mathrm{tr} F^2 .$$

(ii) With the help of Chern-Simons forms, applied to gravity gauge theory (with spin connection gauge field and Lorentz gauge transformations) and Yang-Mills gauge theory (with vector-potential gauge field and local gauge transformation of it), we will get the following composition:

$$Y_4 = d\omega_3, \quad \omega_3 = \omega_{3R} + \omega_{3F},$$

$$\omega_{3F} = -\frac{1}{4} \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad \omega_{3R} = \frac{N+16}{192} \mathrm{tr} \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right) .$$

In general $\mathrm{tr} F^2 = d\omega_3$ therefore, because of $\mathrm{tr} F^2$ is gauge invariant, ω_3 should vary on closed expression. Recalling that $\delta A = d\Lambda + [A, \Lambda]$, one concludes that $\delta\omega_3 = \mathrm{tr}(d\Lambda \wedge dA)$, which may be rewritten as $\delta\omega_3 = d\mathrm{tr}(\Lambda \wedge dA)$ (we can transform it to $\mathrm{tr}(d\Lambda \wedge A)$ via addition of exact form). All together:

$$\delta\omega_3 = dG_2,$$

where

$$G_2 = \frac{N+16}{192}\mathrm{tr}(\Theta \wedge d\omega) - \frac{1}{4}\mathrm{tr}(\Lambda \wedge dA).$$

Problem 5.12

These relations obviously follow from relation

$$\mathrm{Tr}e^{iF} = \frac{1}{2}(\mathrm{tr}\cos F)^2 - \frac{1}{2}\mathrm{tr}\cos 2F,$$

which is supposed to be proved in the solution of Problem 5.10 for $SO(N)$. Comparing different powers of F in r.h.s. and l.h.s. one gets

$$\mathrm{Tr}F^2 = (N-2)\mathrm{tr}F^2,$$

$$\mathrm{Tr}F^4 = (N-8)\mathrm{tr}F^4 + 3(\mathrm{tr}F^2)^2,$$
$$\mathrm{Tr}F^6 = (N-32)\mathrm{tr}F^6 + 15\mathrm{tr}F^2\mathrm{tr}F^4,$$

which in the case of $N = 32$ gives

$$\mathrm{Tr}F^2 = 30\mathrm{tr}F^2,$$
$$\mathrm{Tr}F^4 = 24\mathrm{tr}F^4 + 3(\mathrm{tr}F^2)^2,$$
$$\mathrm{Tr}F^6 = 15\mathrm{tr}F^2\mathrm{tr}F^4,$$

Problem 5.13

What we should check is that equation

$$\mathrm{Tr}F^6 = \frac{1}{48}\mathrm{Tr}F^2\mathrm{Tr}F^4 - \frac{1}{14400}(\mathrm{Tr}F^2)^3 \tag{5.48}$$

is satisfied by E_8 group, and then due to direct sum decomposition of adjoint representation of $E_8 \times E_8$ into independent blocks of E_8's this condition will be satisfied by total $E_8 \times E_8$. For E_8 the following equations take place (see BBS (5.146)):

$$\mathrm{Tr}F^4 = \frac{1}{100}(\mathrm{Tr}F^2)^2, \quad \mathrm{Tr}F^6 = \frac{1}{7200}(\mathrm{Tr}F^2)^3,$$

first of them is proved in the solution of Problem 5.15, and the second must be proved in a similar manner. Substituting these equations into (5.48) we obviously verify its validity.

Problem 5.14

Let's check whether proposed in the statement of this problem gauge groups satisfy conditions of cancellation of anomalies. First condition is that the dimension of gauge group should be 496 is obviously satisfied by both of variants. The second condition is (5.48), where F is any generator of gauge group.

For $U(1)^{496}$ any generator in adjoint representation is a zero matrix, because the group is abelian and any multiplication in algebra (which is commutator) will give zero result. Therefore all traces are zero and condition (5.48) is obviously satisfied.

In the case of $E_8 \times U(1)^{248}$ in fundamental representation there's a decomposition

$$F = \begin{pmatrix} F_E & 0 \\ 0 & F_U \end{pmatrix},$$

where F_E is a E_8 generator and F_U is a $U(1)^{248}$ generator. In adjoint representation it obviously goes to

$$\mathrm{ad}F = \begin{pmatrix} \mathrm{ad}F_E & 0 \\ 0 & 0 \end{pmatrix},$$

and therefore the traces over any power of $U(1)^{248}$ part are zero and have no contribution to the total trace and we can separately study traces over E_8 block. Therefore cancellation of anomalies in this case obviously follows from the cancellation of anomalies for E_8, because condition (5.48) is satisfied by E_8.

`Problem` **5.15**

The basis of E_8 may be constructed out of generators J_{ij} of its $SO(16)$ subgroup and generators Q_α, transforming in a positive chirality representation **128** of $SO(16)$:

$$[J_{ij}, Q_\alpha] = (\sigma_{ij})_{\alpha\beta} Q_\beta,$$

where $\sigma_{ij} = \frac{1}{4}[\gamma_i, \gamma_j]$ are generators of $SO(16)$ in chiral spinor representation. Obviously there're 120 generators of $SO(16)$ and $\frac{1}{2}2^{16/2} = 128$ positive-chirality (half of $Spin(16)$) generators Q_α. In this solution we will focus on proof of first of BBS (5.146) equalities for F being $SO(16)$ generator. Suppose $F = J_{ij}$ is some element of $SO(16)$ subalgebra. Using commutators of E_8 in considered here decomposition we will get the following $\mathbf{248} = \mathbf{120} + \mathbf{128}$ decomposition of adjoint representation of F:

$$\mathrm{ad}\ F = \begin{pmatrix} \mathrm{ad}\ F_{SO(16)} & 0 \\ 0 & (\sigma_F)_{\alpha\beta} \end{pmatrix},$$

where $\sigma_F = \sigma_{ij}$, and ad $F_{SO(16)}$ refer to adjoint representation of F as an element of $SO(16)$, not the whole E_8. For the last value we may use formulae from solution of Problem 5.12:

$$\mathrm{Tr}\ (\mathrm{ad}F_{SO(16)})^2 = 14\mathrm{tr}F^2_{SO(16)} = -28, \quad \mathrm{Tr}\ (\mathrm{ad}F_{SO(16)})^4 = 8\mathrm{tr}F^4_{SO(16)} + 3(\mathrm{tr}F^2_{SO(16)})^2 = 28,$$

where because of $F_{SO(16)}$ is $SO(16)$ generator J_{ij} in fundamental representation, which has only two non-zero elements, thus trace of any even power of it is easily calculated.

The σ part of trace is calculated with the help of equality

$$\sigma_F^2 = -\frac{I}{4},$$

where I is 128×128 matrix. Using this fact will get expressions for total traces:

$$\mathrm{Tr}(\mathrm{ad}F)^2 = -28 - \frac{1}{4}\mathrm{Tr}I = -60, \quad \mathrm{Tr}(\mathrm{ad}F)^4 = 28 + \frac{1}{16}\mathrm{Tr}I = 36.$$

It's obviously that

$$\mathrm{Tr}\ (\mathrm{ad}F)^4 = \frac{(\mathrm{Tr}\ \mathrm{ad}F^2)^2}{100}$$

is satisfied.

6 T-duality and D-branes

`Problem` **6.1**
Let's proceed to superconformal symmetry from world-sheet (space-time) supersymmetry. Superconformal transformations are supersymmetry transformations, written for fields depending on complex conformal coordinates z, $\bar{z}$. For example, we can write down world-sheet supersymmetry transformations in their superconformal form:

$$\delta X^\mu(z,\bar{z}) = \delta X^\mu_L(\bar{z}) + \delta X^\mu_R(z) = -\tilde{\varepsilon}(\bar{z})\tilde{\psi}^\mu(\bar{z}) - \varepsilon(z)\psi^\mu(z),$$
$$\delta\psi^\mu(z) = \varepsilon(z)\partial X^\mu(z), \qquad \delta\tilde{\psi}(\bar{z}) = \tilde{\varepsilon}(\bar{z})\bar{\partial}X^\mu(\bar{z}).$$

World-sheet RNS strings are equivalent to GS strings with $\mathcal{N}=2$ space-time supersymmetry. This means that type-II superstring theories may be formulated either in RNS or GS formalisms. Here we are going to solve the stated problem in terms of RNS formalizm.

Consider T-duality for superstring in $M_{10} = R_{8,1} \times S^1$, i.e. superstring with compactified ninth dimension.

From superconformal transformations to be symmetry of the model it follows that if $X'^9_R(z) = -X^9_R(z)$ in T-dual theory, then we should also define $\psi'^9(z) = -\psi^9(z)$ in T-dual theory. The difference between type-IIA and type-IIB RNS superstrings is the relative chirality of R sector right- and left-movers. If the chiralities are the same, it's type-IIB theory. But for R sector it's known that zero modes of expansion of fermionic field form a representation of Clifford algebra. Therefore reversion $\psi^9(z) \to -\psi^9(z)$ also means $\Gamma^9 \to -\Gamma^9$ and therefore $\Gamma_{11} \to -\Gamma_{11}$. The last obviously leads to the change of chiralities for right-moving sector, which obviously interchanges type-IIA and type-IIB theories.

As for mass-shell conditions, they are not changed, because T-duality besides $R \to \tilde{R} = \alpha'/R$ also assumes interchange of winding number W and Kaluza-Klein number K. The fermionic part in number operators N_L and N_R is not affected by T-duality, while contemplations for bosonic part are exactly the same as in the case of bosonic string.

Therefore there's a correspondence - T-duality - between spectrum of type-IIA superstrings with $X^9 = X^9_R + X^9_L$ compactified on a circle with radius R, and type-IIB superstring with $X'^9 = X^9_L - X^9_R$, compactified on a circle with radius $\tilde{R} = \alpha'/R$.

`Problem` **6.2**
Consider oriented open string with one coordinate X being compactified on a circle of radius R and coupled to $U(N)$ field A by term

$$\langle ij|L_{int}|ij\rangle = \dot{X}\left(\langle i|A|i\rangle - \langle j|A|j\rangle\right). \tag{6.49}$$

The idea behind such coupling is that both ends of the string form a representation space for gauge group $U(N)$, therefore both ends of this string possess $U(N)$ charge, and again therefore they both interact with field A. But they form conjugate representations N and $\bar{N}$ of $U(N)$, which means that they have $U(N)$ charges of opposite signs $+1$ and -1 (and thus transformed as $|i\rangle \to e^{iU}|i\rangle$ and $|j\rangle \to e^{-iU}|j\rangle$). But coupling term is proportional to charge. E.g., Maxwell field it's $e\dot{X}A$. The physical sense is that charges of opposite signs interact with the same external field by forces of opposite directions. This is exactly what was used for coupling Lagrangian (6.49).

The state of string may be described by vertex operator, which anyway contains a factor e^{iPX}. This should be single-valued, which requires Kaluza-Klein quantization $P = \frac{K}{R}$. Total momentum P of string in compactified direction is a canonical momentum composed of string momentum p and shift (for a string in $|ij\rangle$ state)

$$\langle ij|\frac{\partial L_{int}}{\partial \dot{X}}|ij\rangle = \langle i|A|i\rangle - \langle j|A|j\rangle \tag{6.50}$$

due to interaction with external field A. If $A = \frac{1}{2\pi R}\mathrm{diag}\{\theta_1, \dots \theta_N\}$, string momentum in the state $|ij\rangle$ is given by

$$p = \frac{K}{R} - \frac{\theta_i - \theta_j}{2\pi R}.$$

Formula BBS (6.38) is a trivial consequence of the last equation.

`Problem` **6.3**

(i) Let's reformulate the problem as follows. Consider now closed string vacuum state. It's fermionic part is a product of right- and left-moving GS spinors ψ_R, ψ_L. From the position of space-time supersymmetry this assumes $\mathcal{N} = 2$ supersymmetry, and corresponding string theory is called type-II superstring.

Define $\Gamma_{11} = \Gamma_0 \cdots \Gamma_9$. As soon as in Majorana representation hermitian conjugation means a transposing, therefore Γ_0 is antisymmetric, while Γ_i are symmetric. It follows also that Γ_{11} is symmetric.

Spinors of left- and right-moving sectors are assumed to have definite chirality: it's assumed $\Gamma_{11}\psi_R = \psi_R$, $\Gamma_{11}\psi_L = \xi\psi_L$, where $\xi = -1$ for type-IIA superstring theory and $\xi = 1$ for type IIB theory.

Define a 32×32 matrix F with elements

$$F_{\alpha\beta} = \psi_{R\alpha}\Gamma^0_{\beta\gamma}\psi_{L\gamma}.$$

It follows that

$$(\Gamma_{11}F)_{\alpha\beta} = \Gamma_{11\alpha\gamma}F_{\gamma\beta} = (\Gamma_{11\alpha\gamma}\psi_{R\gamma})\Gamma^0_{\beta\delta}\psi_{L\delta} = \psi_{R\alpha}\Gamma^0_{\beta\gamma}\psi_{L\gamma} = F_{\alpha\beta},$$

$$(F\Gamma_{11})_{\alpha\beta} = F_{\alpha\gamma}\Gamma_{11\gamma\beta} = \psi_{R\alpha}\Gamma^0_{\gamma\delta}\psi_{L\delta}\Gamma_{11\gamma\beta} = -\psi_{R\alpha}\Gamma^0_{\delta\gamma}\Gamma_{11\gamma\beta}\psi_{L\delta} = \psi_{R\alpha}\Gamma_{11\delta\gamma}\Gamma^0_{\gamma\beta}\psi_{L\delta} =$$
$$= \psi_{R\alpha}\Gamma^0_{\gamma\beta}\Gamma_{11\gamma\delta}\psi_{L\delta} = \xi\psi_{R\alpha}\Gamma^0_{\gamma\beta}\psi_{L\gamma} = -\xi\psi_{R\alpha}\Gamma^0_{\beta\gamma}\psi_{L\gamma} = -\xi F_{\alpha\beta}.$$

If we employ an expansion

$$F = \sum_{k=0}^{10} \frac{1}{k!}F_{\mu_1\cdots\mu_k}\Gamma^{\mu_1\cdots\mu_k}$$

and identities

$$\Gamma_{11}\Gamma^{\mu_1\cdots\mu_k} = \frac{(-1)^{[k/2]}}{(10-k)!}\varepsilon^{\mu_1\cdots\mu_{10}}\Gamma_{\mu_{k+1}\cdots\mu_{10}},$$

$$\Gamma^{\mu_1\cdots\mu_k}\Gamma_{11} = \frac{(-1)^{[(k+1)/2]}}{(10-k)!}\varepsilon^{\mu_1\cdots\mu_{10}}\Gamma_{\mu_{k+1}\cdots\mu_{10}},$$

which may be checked directly on simple examples, we will be able to rewrite derived $\Gamma_{11}F = F$, $F\Gamma_{11} = -\xi F$ as

$$F^{\mu_1\cdots\mu_k} = \frac{(-1)^{[k/2]}}{(10-k)!}\varepsilon^{\mu_1\cdots\mu_{10}}F_{\mu_{k+1}\cdots\mu_{10}}, \tag{6.51}$$

$$F^{\mu_1\cdots\mu_k} = \frac{(-1)^{[(k+1)/2]+1}}{(10-k)!}\xi\varepsilon^{\mu_1\cdots\mu_{10}}F_{\mu_{k+1}\cdots\mu_{10}}. \tag{6.52}$$

For type IIB theory compatibility of the last two expressions requires odd k-values, while type IIA theory reqires even k values.
(ii) For odd k values and type-IIB theory or for even k values and type-IIA theory equations (6.51) and (6.52) are equivalent to each other and has the form of

$$F^{\mu_1\cdots\mu_k} = \frac{(-1)^{[k/2]}}{(10-k)!}\varepsilon^{\mu_1\cdots\mu_{10}}F_{\mu_{k+1}\cdots\mu_{10}}.$$

Remember that in the theory with flat space-time background (the last is used to substitute $\sqrt{-g}$ into a general formula):

$$(\star F)^{\mu_1\cdots\mu_k} = \frac{1}{(10-k)!}\varepsilon^{\mu_1\cdots\mu_{10}}F_{\mu_{k+1}\cdots\mu_{10}}.$$

Comparing this formulae we conclude that

$$(\star F)_k = (-1)^{[k/2]}F_{10-k}.$$

From this equation it follows that we must consider only field strength tensors with rank $k < 6$, because higher ranks are expressible through them via Poincare duality. Tensor F_5 is dual to itself. (iii) The calculation of number of independent degrees of freedom in different type string theories gives:

$$IIA: \quad 1 + \frac{10\cdot 9}{2!} + \frac{10\cdot 9\cdot 8\cdot 7}{4!} = 256, \tag{6.53}$$

$$IIB: \quad 10 + \frac{10\cdot 9\cdot 8}{3!} + \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{2\cdot 5!} = 256, \tag{6.54}$$

where we've divided by 2 in the r.h.s. of the last equation, because in difference with the cases of other forms, the form $F_{(5)}$ is restricted by a condition of self-duality - analog of real-value condition on number - which follows from preceding point of this solution. The number 256 is equal to the number of independent product results of two Majorana-Weyl spinors in ten dimensions. But obviously this relates to GS superstrings only before κ-gauge (light cone κ-gauge choice). Remember, that without gauging away half of MW fermions with the help of κ-symmetry (i.e. without κ-gauge) the number of fermionic degrees of freedom is incorrect even from the point of view of equality of total numbers of fermionic and bosonic degrees of freedom.

Problem 6.4

To solve this problem one needs the following identities, which may be verified directly on simple examples:

$$\Gamma^\mu\Gamma^{\nu_1\cdots\nu_k} = \Gamma^{\mu\nu_1\cdots\nu_k} - \frac{1}{(k-1)!}\eta^{\mu[\nu_1}\Gamma^{\nu_2\cdots\nu_k]}, \tag{6.55}$$

$$\Gamma^{\nu_1\cdots\nu_k}\Gamma^{\mu} = \Gamma^{\nu_1\cdots\nu_k\mu} - \frac{1}{(k-1)!}\eta^{\mu[\nu_k}\Gamma^{\nu_1\cdots\nu_{k-1}]} \tag{6.56}$$

The matrix field F introduced in the solution of previous problem, is composed appropriately out off Dirac massless spinors, and therefore satisfy massless Dirac equation:

$$p_\mu\Gamma^\mu F = F\Gamma^\mu p_\mu = 0,$$

from which with the help of identities (6.55) and (6.56) one can get

$$p_\mu\Gamma^\mu F = p_\mu\sum_{k=0}^{10}\frac{1}{k!}F_{\mu_1\cdots\mu_k}\Gamma^{\mu\mu_1\cdots\mu_k} - p_\mu\sum_{k=0}^{10}\frac{1}{k!(k-1)!}F_{\mu_1\cdots\mu_k}\eta^{\mu[\mu_1}\Gamma^{\mu_2\cdots\mu_k]} = 0, \tag{6.57}$$

$$F\Gamma^\mu p_\mu = p_\mu\sum_{k=0}^{10}\frac{1}{k!}F_{\mu_1\cdots\mu_k}\Gamma^{\mu_1\cdots\mu_k\mu} - p_\mu\sum_{k=0}^{10}\frac{1}{k!(k-1)!}F_{\mu_1\cdots\mu_k}\eta^{\mu[\mu_k}\Gamma^{\mu_1\cdots\mu_{k-1}]} = 0. \tag{6.58}$$

From (6.58) one can easily go to

$$p_\mu\Gamma^\mu F = p_\mu\sum_{k=0}^{10}\frac{1}{k!}(-1)^k F_{\mu_1\cdots\mu_k}\Gamma^{\mu\mu_1\cdots\mu_k} + p_\mu\sum_{k=0}^{10}\frac{1}{k!(k-1)!}(-1)^k F_{\mu_1\cdots\mu_k}\eta^{\mu[\mu_1}\Gamma^{\mu_2\cdots\mu_k]} = 0. \tag{6.59}$$

As soon as all k values are either odd or even, then we can cancel $(-1)^k$ multiplier. If we don't concretize type-IIA or type-IIB theory we deal with, then we still write summation over all k, keeping in mind all features of such notation which we have covered above. Adding and subtracting (6.57) and (6.59) one can achieve

$$p_\mu\sum_{k=0}^{10}\frac{1}{k!}F_{\mu_1\cdots\mu_k}\Gamma^{\mu\mu_1\cdots\mu_k} = 0,$$

$$p_\mu\sum_{k=0}^{10}\frac{1}{k!(k-1)!}F_{\mu_1\cdots\mu_k}\eta^{\mu[\mu_1}\Gamma^{\mu_2\cdots\mu_k]} = 0,$$

which may be rewritten as

$$\sum_{k=0}^{10}\frac{1}{(k-1)!}p_{[\mu}F_{\mu_1\cdots\mu_k]}\Gamma^{\mu\mu_1\cdots\mu_k} = 0,$$

$$\sum_{k=0}^{10}\frac{1}{((k-1)!)^2}p^{\mu_1}F_{\mu_1\cdots\mu_k}\Gamma^{\mu_2\cdots\mu_k} = 0.$$

In the second equality we took index μ_1 out of explicit antisymmetrization (it's still implicitly antisymmetrized with other indices due to contraction of the whole construction with F-components), which added factor k corresponding to the number of different positions of this index. Due to independence of terms being elements of Clifford algebra $Cl_{9,1}$ one goes to

$$p_{[\mu}F_{\mu_1\cdots\mu_k]} = 0,$$

$$p^{\mu_1}F_{\mu_1\cdots\mu_k} = 0.$$

These equations may be rewritten in coordinate repesentation which has the form stated in the condition of the problem.

In the point (ii) in the solution of Problem 6.3 we have figured out that $F_n = \pm \star F_{10-n}$. As soon as field equations and Bianchi identities may be rewritten in differential form notation as

$$d \star F_n = 0, \quad dF_n = 0,$$

and due to $\star\star = \pm 1$, then we have correspondingly

$$dF_{10-n} = 0, \quad d \star F_n = 0.$$

Problem **6.5**

For shortness of notation define Lagrangian in BBS (6.94) through the expression (without dilation term BBS (6.93)):

$$4\pi\alpha' S = \int d^2\sigma L,$$

which will lead to

$$\frac{\partial L}{\partial(\partial_\alpha V_\beta)} = \varepsilon^{\alpha\beta}\tilde{X}^9, \tag{6.60}$$

$$\frac{\partial L}{\partial V_\beta} = -2g_{99}\sqrt{-h}h^{\alpha\beta}V_\alpha - 2g_{9\mu}\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu + \varepsilon^{\beta\alpha}B_{9\mu}\partial_\alpha X^\mu, \tag{6.61}$$

which is to be used for Lagrange equations for field V^β.

There's a subtlety in this problem which should be underlined. For the construction of action we should use invariant measure $d^2\sigma\sqrt{-h}$ on world-sheet. But we can use $\varepsilon^{\alpha\beta}$-symbols (they are not tensors, by the way, on curved world-sheet) instead of $\sqrt{-h}$. The thing is that $E^{\alpha\beta} = \frac{\varepsilon^{\alpha\beta}}{\sqrt{-h}}$ is a tensor, and therefore we may contemplate in the following way: $\int d^2\sigma\varepsilon^{\alpha\beta}\Omega_{\alpha\beta} = \int d^2\sigma\sqrt{-h}E^{\mu\nu}\Omega_{\mu\nu}$. The later integral is built of invariant measure $d^2\sigma\sqrt{-h}$ and tensor product $E^{\mu\nu}\Omega_{\mu\nu}$. Both values are invariants and the action is thus invariant too.

The second subtlety is that while we can use metric tensor to lower indices of $E^{\alpha\beta}$ tensor, we can't do the same with non-tensor value $\varepsilon^{\alpha\beta}$. Therefore we can't say that $E_{\alpha\beta}$ is given by $\frac{\varepsilon_{\alpha\beta}}{\sqrt{-h}}$. We can't say, e.g., that $\varepsilon^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X$ is equal to $\varepsilon_{\alpha\beta}\partial^\alpha X \cdot \partial^\beta X$. We must do it the following way:

$$\varepsilon^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X = \sqrt{-h}E^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X = \sqrt{-h}E_{\alpha\beta}\partial^\alpha X \cdot \partial^\beta X.$$

We also know how antisymmetric tensor with lower indices looks like: $E_{\alpha\beta} = \sqrt{-h}\varepsilon_{\alpha\beta}$.

Using formulae (6.60) and (6.61) for Lagrange equation on V-field we will get:

$$V^\beta = -\frac{E^{\alpha\beta}}{2g_{99}}\partial_\alpha\tilde{X}^9 - \frac{g_{9\mu}}{g_{99}}\partial^\beta X^\mu + \frac{E^{\beta\alpha}}{2g_{99}}B_{9\mu}\partial_\alpha X^\mu.$$

What is rest is to substitute this formula inside the action BBS (6.94), paying attention on noted above, and get an answer:

$$\tilde{g}_{99} = \frac{1}{g_{99}}, \quad \tilde{g}_{9\mu} = \frac{B_{9\mu}}{g_{99}}, \quad \tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{B_{9\mu}B_{9\nu} - g_{9\mu}g_{9\nu}}{g_{99}},$$

$$\tilde{B}_{9\mu} = \frac{g_{9\mu}}{g_{99}}, \quad \tilde{B}_{\mu\nu} = B_{\mu\nu} + \frac{g_{9\mu}B_{9\nu} - B_{9\mu}g_{9\nu}}{g_{99}}.$$

Problem 6.6

First we are going to explore representation of delta-function in spherical symmetric case. Poisson equation for unit charge with potential $\phi = \frac{1}{r}$ looks like

$$\Delta\frac{1}{r} = -4\pi\rho,$$

where $\int d^3x\rho = 4\pi\int drr^2\rho(r) = 1$ is a total charge. Therefore we conclude that

$$\rho(r) = \frac{\delta(r)}{4\pi r^2},$$

hence

$$\delta(r) = -r^2\Delta\frac{1}{r}.$$

Due to the fact that spherical symmetric Laplacian acts as $\Delta f(r) = \frac{1}{r^2}\partial_r\left(r^2\partial_r f(r)\right)$ we obtain formal generalized function representation

$$\delta(r) = \partial_r \cdot 1.$$

The representation of delta function constructed here is useful for expression for current $j^\mu = \rho u^\mu$.

Total action is given by

$$S = S_{BI} + S_P,$$

where Born-Infeld action is (in the spherical symmetric case after explicit integration over angular coordinates, constant $\frac{1}{k^2}$ with $k = 2\pi\alpha'$ is introduced to make action dimensionless)

$$S_{BI} = \frac{4\pi}{k^2}\int d\tau drr^2\left(1 - \sqrt{-\det(\eta_{\alpha\beta} + kF_{\alpha\beta})}\right), \tag{6.62}$$

and particle action looks like

$$S_P = -\int d\tau drA^\mu j_\mu. \tag{6.63}$$

We consider the case of particle in its rest frame at coordinate $\mathbf{x} = 0$, thus 4-velocity is equal to $u_\mu = (-1, 0, 0, 0)$ and current is given by $j_\mu = e\delta(r)u_\mu$, which will reduce S_P to its more familiar form

$$S_P = -e\int d\tau u_\mu A^\mu.$$

For our purposes more appropriate form of action is (6.63), because together with (6.62) it allows to write (τ, r) Lagrangian density L. We search for Coulomb potential $A^\alpha = (\phi(r), 0, 0, 0)$ with the sole independent non-zero component of field strength $F_{tr} = \phi'(r)$. Lagrangian density is then equal to

$$L = \frac{4\pi r^2}{k^2}\left(1 - \sqrt{1 - k^2(\phi')^2}\right) + e\phi(r)\delta(r).$$

Lagrange equation obviously gives us

$$4\pi\partial_r\left(\frac{r^2\phi'}{\sqrt{1-k^2(\phi')^2}}\right)=e\delta(r),$$

using δ-function representation constructed above we proceed to (use also $E_r=-\phi'(r)$):

$$\frac{r^2E_r}{\sqrt{1-k^2E_r^2}}=e,$$

from which it follows the solution

$$E_r=\frac{e}{\sqrt{r^4+r_0^4}},$$

with $r_0^2=ek$ (remember that charge is dimensionless and $[k]=[\alpha']=2[L]$).

We may use Noether theorem to construct T^{00} component of energy-momentum tensor for field part of the action (i.e. for Born-Infeld part). Here it's $T^{00}=-L_{BI}=\frac{4\pi r^2}{k^2\sqrt{1-k^2E_r^2}}-\frac{4\pi r^2}{k^2}$ (because all time derivatives are zero and there's no momentum which would take part in formula for T^{00}), which is a density of energy in coordinate r, and on-shell this is given by $T^{00}=\frac{4\pi r^4}{k^2\sqrt{r^4+r_0^4}}-\frac{4\pi r^2}{k^2}$. Total energy is therefore $\varepsilon=\int drT^{00}=\int dr(\frac{4\pi r^4}{k^2\sqrt{r^4+r_0^4}}-\frac{4\pi r^2}{k^2})$. The value of integrand is finite when $r\to 0$ in difference with divergent $\sim\frac{1}{r^2}$ in Maxwell theory and has asymptotic behavior $\sim\frac{1}{r^2}$ as $r\to\infty$, therefore integral is convergent.

`Problem` **6.7**

(i) Lagrangian density is given by

$$L=-T_{Dp}\sqrt{-\det H_{\alpha\beta}}=-T_{Dp}\sqrt{-H},$$

with $H_{\alpha\beta}=G_{\alpha\beta}+kb_{\alpha\beta}+kF_{\alpha\beta}=G_{\alpha\beta}+k\mathcal{F}_{\alpha\beta}$. One can easily find

$$\frac{\partial L}{\partial(\partial_\alpha A_\beta)}=\frac{\partial L}{\partial F_{\gamma\delta}}\frac{\partial F_{\gamma\delta}}{\partial(\partial_\alpha A_\beta)}=k\frac{\partial L}{\partial H_{\gamma\delta}}\frac{\partial F_{\gamma\delta}}{\partial(\partial_\alpha A_\beta)}=\frac{k}{2}T_{Dp}\sqrt{-H}H^{\gamma\delta}(\delta^\alpha_\gamma\delta^\beta_\delta-\delta^\alpha_\delta\delta^\beta_\gamma)=kT_{Dp}\sqrt{-H}H^{\alpha\beta},$$

which is to be used for construction of gauge field equations of motion:

$$\partial_\alpha(\sqrt{-H}H^{\alpha\beta})=0.$$

(ii) In the solution of Exercise 6.9 (page 243) it's found that the main Lagrangian constituent $\sqrt{-H}$ is expanded as

$$\sqrt{-H}=\sqrt{-G}\left(1+\frac{k^2}{4}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}+O(k^4)\right).$$

The value of G is k-independent, therefore in the leading order of k we obtain the following equation, correcting Maxwell:

$$\partial_\alpha(\sqrt{-G}H^{\alpha\beta})+\frac{k^2}{4}\partial_\alpha(\sqrt{-G}\mathcal{F}^2G^{\alpha\beta})=0.$$

`Problem` **6.9**

(i) Type-I' theory is constructed as orientifold projection of type-IIA superstring. Presence of

$D8$-branes assumes performed duality in X^9 direction (starting from all-Neumann string) and therefore compactification of this direction. If R is radius of compactification (in Neumann-boundary open string theory), then in the T-dual theory D8 branes emerges perpendicular to $\tilde{X}^9$, where compactification radius of later is $\tilde{R} = \alpha'/R$.

Orientifold projection may be interpreted with the help of introducing of orientifold planes. Here orientifold 8-planes (8 for 8 non-compact spatial dimensions) are located at singular points $\tilde{X} = 0$, $\pi\tilde{R}$ of BBS (6.83) orbifolding.

If D-branes are located at some points $\tilde{X}_L = \theta_L\tilde{R}$ and $\tilde{X}_R = \theta_R\tilde{R}$ (N_1 and $N_2 = 16 - N_1$ branes) in the interior of $(0, \pi\tilde{R})$ interval, then gauge symmetry is broken to $U(N_1)\times U(16-N_1)$. If $\theta_L = 0$, i.e. N_1 $D8$-branes are located at $\tilde{X} = 0$, then they give unbroken $SO(2N_1) \times U(1)$ gauge group. If $\theta_R = \pi$, i.e. N_2 $D8$-branes are located at $\tilde{X} = \pi\tilde{R}$, then they give unbroken $SO(2N_2)\times U(1)$ gauge group. Gauge symmetry in the first-non-second case is $SO(2N_1)\times U(1)\times U(16 - N_1)$, in the second-non-first case is $U(N_1) \times SO(32 - 2N_1) \times U(1)$. If $\theta_L = 0$, $\theta_R = \pi\tilde{R}$, then gauge symmetry is maximal: $SO(2N_1) \times SO(32 - 2N_1) \times U(1)^2$.

(ii) According to previous point this is the case of $\theta'_L = 0$, $\theta'_R = \pi\tilde{R}$ and gauge group $SO(16) \times U(1) \times SO(16) \times U(1)$.

`Problem` **6.11**

(i) We are to determine variation of 2-form

$$b = (\bar{\Theta}^1\Gamma_\mu d\Theta^1 - \bar{\Theta}^2\Gamma_\mu d\Theta^2)(dX^\mu - \frac{1}{2}\bar{\Theta}^A\Gamma^\mu d\Theta^A),$$

where wedge product is assuemd implicitly, under supersymmetry transformations:

$$\delta\Theta^A = \varepsilon^A, \qquad \delta X^\mu = \bar{\varepsilon}^A\Gamma^\mu\Theta^A.$$

This is a straightforward task. What is important is that due to the actual presence of wedge product between differentials (basic Grassmann 1-forms), the values of the type $\bar{\varepsilon}^1\Gamma_\mu d\Theta^1$ and $\bar{\Theta}^2\Gamma^\mu d\Theta^2$ anticommute, being in wedge product. With respect to this feature we will result in

$$\delta b = d((\bar{\varepsilon}^1\Gamma_\mu d\Theta^1 - \bar{\varepsilon}^2\Gamma_\mu d\Theta^2)X^\mu) - \bar{\varepsilon}^1\Gamma_\mu d\Theta^1\bar{\Theta}^1\Gamma^\mu d\Theta^1 + \bar{\varepsilon}^2\Gamma_\mu d\Theta^2\bar{\Theta}^2\Gamma^\mu d\Theta^2.$$

The last two tems vanish, because they are of the type, described in Problem 5.7. Therefore

$$\delta b = -d\Lambda,$$

where

$$\Lambda = (\bar{\varepsilon}^2\Gamma_\mu d\Theta^2 - \bar{\varepsilon}^1\Gamma_\mu d\Theta^1)X^\mu.$$

(ii) Requiring the value of 2-form $\mathcal{F} = F + b$ to be supersymmetric, where $F = dA$, one obtains the following law for A-field transformation (up to an exact form, i.e. gauge transformation):

$$\delta A = \Lambda.$$

`Problem` **6.12**

Define tensor components:

$$G_{\alpha\beta} = g_{\alpha\beta} + B_{\alpha\beta} + k^2\partial_\alpha\Phi^i\partial_\beta\Phi^i + kF_{\alpha\beta}$$

and its determinant G, which is obtained by pullback of NS-NS background fields $g_{\mu\nu}$, $B_{\mu\nu}$ and Maxwell field $F_{\mu\nu}$ to the world-volume of Dp-brane. Obviously then gauge variation (BBS (6.130) with appropriate variation of Maxwell field) of DBI action is given by

$$\delta S_{Dp} = -\frac{T_{Dp}}{2}\int d^{p+1}\sigma e^{-\Phi_0}\sqrt{-G}G^{\alpha\beta}\delta G_{\alpha\beta} = -\frac{T_{Dp}}{2}\int d^{p+1}\sigma e^{-\Phi_0}\sqrt{-G}G^{\alpha\beta}(\delta B_{\alpha\beta} + k\delta F_{\alpha\beta}).$$

If we require it to be zero, i.e. we require gauge invariance of the action, then we must require

$$\delta B_{\alpha\beta} + k\delta F_{\alpha\beta} = (\delta B_{\mu\nu} + k\delta F_{\mu\nu})\partial_\alpha X^\mu \partial_\beta X^\nu = 0,$$

which in differential form notation for $\delta B = d\Lambda$ assumes $\delta F = -\frac{1}{k}d\Lambda$, therefore $\delta A = -\frac{1}{k}\Lambda$.

`Problem` **6.13**

(i) We are going to exploit general formula of DBI D-brane static action

$$S_{Dp} = -T_{Dp}\int d^{p+1}\sigma\sqrt{-G}$$

with G being determinant of tensor matrix with elements

$$G_{\alpha\beta} = \eta_{\alpha\beta} + k^2\partial_\alpha\Phi^i\partial_\beta\Phi^i + kF_{\alpha\beta}.$$

Here $p = 3$, we choose coordinates t, r, θ, ϕ (we don't need to insert Jacobian factor for this coordinates because of tensor determinant is present, and it's written in the same coordinates as the measure $d^{p+1}\sigma$), there's only one non-zero scalar field $\Phi(r)$ and only one Maxwell field component $A_t(r)$. Therefore

$$G_{\alpha\beta} = \begin{bmatrix} -1 & kA_t'(r) & 0 & 0 \\ -kA_t'(r) & 1+k^2(\Phi'(r))^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{\alpha\beta}$$

From this we can easily figure out determinant G and the whole action

$$S_{D3} = -T_{D3}\int dtdrd\theta d\phi\sqrt{1 + k^2(\Phi'(r))^2 - k^2(A_t'(r))^2}. \tag{6.64}$$

(ii) Free (we don't consider coupling of $D3$-brane to gauge fields) equations of motion which obviously follow from action (6.64) are

$$\partial_r\left(\frac{\Phi'}{\sqrt{1 + k^2\Phi'^2 - k^2A_t'^2}}\right) = 0, \quad \partial_r\left(\frac{A_t'}{\sqrt{1 + k^2\Phi'^2 - k^2A_t'^2}}\right) = 0.$$

Integration of this system of differential equations will give free solution with no sources:

$$A_t = C_1 r + A_t(0), \quad \Phi(r) = C_2 r + \Phi(0).$$

(iii) If there's an electric charge at the origin, then our way of solution should be similar to that in the solution of Problem 6.6. We add Lagrangian describing coupling of point particle at rest at the origin to Maxwell field F and use δ-function representation, employed there. All this will lead us to the following equations of motion

$$\partial_r \left(\frac{\Phi'}{\sqrt{1 + k^2 \Phi'^2 - k^2 A_t'^2}} \right) = 0, \quad \frac{A_t'}{\sqrt{1 + k^2 \Phi'^2 - k^2 A_t'^2}} = e.$$

Here e is dimensionless charge times $\sim \frac{1}{l_s^2}$ - to enable dimension of r.h.s. equals to dimension of l.h.s. The first equation may be integrated as in the previous point (C is a constant):

$$\frac{\Phi'}{\sqrt{1 + k^2 \Phi'^2 - k^2 A_t'^2}} = \frac{1}{C}$$

This obviously gives us

$$\Phi'^2 = \frac{1 - k^2 A_t'^2}{C^2 - k^2}, \tag{6.65}$$

which is to be substituted into equation for A_t. This will give some constant value for A_t' and hence some constant value for Φ'. The type of solution is therefore the same as described at the end of previous point - linear dependence.

To evaluate the range of distances r for which DBI approximation works note that the reason why it can't work is that (according to BBS (6.115)) tension of $D3$-brane is given by

$$T_{D3} = \frac{1}{g_s (2\pi)^3 \alpha'^2},$$

which depicts a divergent growth as $g_s \to 0$. Therefore we must restrict g_s - closed string coupling - that is restrict maximal energy of interaction, or restrict minimal distance r between interacting objects (not just some abstract r). If E is the value of energy cut-off, then $\frac{1}{E}$ is a value of distance cut-off. The interpretation of this formula is that if we use DBI approximation it means, that we can't describe space sharper than $\frac{1}{E}$.

`Problem` **6.14**

(i) Tensor matrix elements in this case are:

$$G_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1+k^2(\Phi'(r))^2 & 0 & 0 \\ 0 & 0 & 1 & \partial_\theta A_\phi \\ 0 & 0 & -\partial_\theta A_\phi & 1 \end{bmatrix}_{\alpha\beta}$$

Therefore action is

$$S_{D3} = -T_{D3} \int dt dr d\theta d\phi \sqrt{(1 + k^2 (\Phi')^2)(1 + (\partial_\theta A_\phi)^2)}.$$

(ii) Free equations of motion are

$$\partial_\theta \left(\frac{\partial_\theta A_\phi}{\sqrt{(1 + k^2 (\Phi')^2)(1 + (\partial_\theta A_\phi)^2)}} \right) = 0, \quad \partial_\theta \left(\frac{\Phi'}{\sqrt{(1 + k^2 (\Phi')^2)(1 + (\partial_\theta A_\phi)^2)}} \right) = 0.$$

We can denote $U = \partial_\theta A_\phi$ and $V = \Phi'$, use equations of motion to find ratio U/V and U, V themselves.

Problem 6.15
Potential

$$V(\Phi) = -\frac{1}{4}\mathrm{Tr}([\Phi^i, \Phi^j][\Phi^i, \Phi^j]) - \frac{if}{3}\varepsilon_{ijk}\mathrm{Tr}(\Phi^i\Phi^j\Phi^k).$$

is extremal under condition

$$[[\Phi^i, \Phi^j], \Phi^j] + if\varepsilon_{ijk}[\Phi^j, \Phi^k] = 0.$$

This is solved for $SU(2)$ group by letting $\Phi^i = \frac{f\alpha^i}{2}$, where α^i is N-dimensional representation of $SU(2)$. If it's irreducible, then

$$\mathrm{Tr}(\alpha^i\alpha^j) = \frac{1}{3}N(N^2-1)\delta^{ij}.$$

Using this formula one computes the value of $V(\Phi)$ for $\Phi^i = \frac{f\alpha^i}{2}$:

$$V(\Phi) = -\frac{f^4}{24}\mathrm{Tr}(\alpha^k\alpha^k) = -\frac{f^4}{72}N^2(N^2-1).$$

For reducible representation $\mathbf{N} = \mathbf{M} + \mathbf{K}$, or $r = r_1 \oplus r_2$ one has

$$\mathrm{Tr}_{r_1\oplus r_2}(\alpha^i\alpha^j) = \mathrm{Tr}_{r_1}(\alpha^i\alpha^j) + \mathrm{Tr}_{r_2}(\alpha^i\alpha^j),$$

and therefore

$$\mathrm{Tr}(\alpha^i\alpha^j) = \frac{1}{3}\left(M(M^2-1) + K(K^2-1)\right)\delta^{ij}.$$

Thus

$$V(\Phi) = -\frac{f^4}{24}\mathrm{Tr}(\alpha^k\alpha^k) = -\frac{f^4}{72}\left(M^2(M^2-1) + K^2(K^2-1)\right).$$

This is higher by its absolute value than that in the case of irreducible representation.

Fuzzy sphere is given by equation

$$\sum_i \langle (X^i)^2 \rangle = R^2,$$

where

$$\langle (X^i)^2 \rangle = \frac{1}{N^2}(2\pi\alpha')^2\mathrm{Tr}[(\Phi^i)^2],$$

and therefore radius of fuzzy sphere in this case is

$$R^2 = \frac{M^2(M^2-1) + K^2(K^2-1)}{3N^2}(\pi\alpha' f)^2.$$

7 Heterotic string

Problem 7.2
Compactification of $26d$ bosonic string theory in this problem is performed by replacement of 16 spatial bosonic coordinates by 32 Majorana fermions. This procedure will preserve the value

$26 = 10 + \frac{32}{2}$ of bosonic string central charge and won't lead to conformal anomaly. The action of such a theory is built in a manner of world-sheet supersymmetric action, despite now, of course, there's no susy here: the number of introduced fermions do not coincide with the number of survived bosons, which prevents possibility of writing world-sheet susy transformations. Fermions form a representation space of $SO(32)$, and because of in conformal notation the action looks like

$$S = \frac{1}{2\pi} \int d^2z \left(2\partial X^\mu \bar{\partial} X_\mu + \frac{1}{2}\lambda^A \bar{\partial}\lambda^A + \frac{1}{2}\tilde{\lambda}^A \partial \tilde{\lambda}^A \right),$$

there's actually a symmetry $SO(32)_L \times SO(32)_R$, transforming left- and right-moving (holomorphic and antiholomorphic) fermions independently. Consider, e.g., how action is transformed under $\delta\lambda_A = \varepsilon_a T^a_{AB}\lambda_B$, $\delta\tilde{\lambda}_A = 0$:

$$\delta S = \delta S_f = \frac{1}{4\pi} \int d^2z \left(\varepsilon_a T^a_{AB} \lambda^B \bar{\partial}\lambda^A \lambda^A (\bar{\partial}\varepsilon_a) T^a_{AB}\lambda^B + \varepsilon_a T^a_{AB}\lambda^A \bar{\partial}\lambda^B \right).$$

Now if we go on-shell, we employ holomorphic property of fermionic field λ_A: $\bar{\partial}\lambda^A = 0$. From the other side variation of action is equal to

$$\delta S = \frac{1}{2\pi} \int d^2z (\bar{\partial}\varepsilon_a) J^a(z),$$

where we've chosen convenient normalization of Noether current. As a result we conclude that

$$J^a(z) = \frac{1}{2} T^a_{AB} \lambda^A(z)\lambda^B(z).$$

Similar arguments lead to the same expression for $SO(32)$ Noether current for symmetry realized on antiholomorphic fermions $\tilde{\lambda}^A$.

It may be easily verified (see solution to Ex. 7.1), that if $[T^a, T^b] = 2if^{abc}T^c$, then Noether currents satisfy OPE for Kac-Moody (current) algebra with level $k = 1$:

$$J^a(z)J^b(w) = \frac{\delta^{ab}}{2(z-w)^2} + i\frac{f^{abc}}{z-w}J^c(w) + \dots .$$

Note, that central charge is $c = \frac{k \, \dim SO(32)}{k + (32-2)} = 16$, as it should be.

We have the following well-known bosonic string mass formulae ($\alpha' = \frac{1}{2}$):
open string: $\frac{1}{2}M^2 = N - a$;
closed string: $\frac{1}{8}M^2 = N_R - a = N_L - a$.
We may choose either periodic (P) boundary conditions for fermions of closed string, which corresponds to $\lambda^A|_{\sigma=\pi} = \tilde{\lambda}^A|_{\sigma=\pi}$ open string boundary condition. Or we may choose antiperiodic (A) closed string boundary conditions for fermions and boundary condition $\lambda^A|_{\sigma=\pi} = -\tilde{\lambda}^A|_{\sigma=\pi}$ for open string. Zero point energies in these cases are:

$$a_P = \frac{8}{24} - \frac{32}{24} = -1,$$

$$a_A = \frac{8}{24} + \frac{32}{24} = 1.$$

"Number" operators are:
open string, P: $N = \sum_{n=1}^{\infty} (\alpha^i_{-n}\alpha^i_n + n\lambda^A_{-n}\lambda^A_n)$,

open string, A: $N = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r\lambda_{-r}^A \lambda_r^A$,

closed string, P: $N_R = \sum_{n=1}^{\infty} (\alpha_{-n}^i \alpha_n^i + n\lambda_{-n}^A \lambda_n^A)$, $\quad N_L = \sum_{n=1}^{\infty} (\tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + n\tilde{\lambda}_{-n}^A \tilde{\lambda}_n^A)$,

closed string, A: $N_R = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r\lambda_{-r}^A \lambda_r^A$, $\quad N_L = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \sum_{r=1/2}^{\infty} r\tilde{\lambda}_{-r}^A \tilde{\lambda}_r^A$.

Summarize results for opens strings:

$$P: \quad \frac{1}{2}M^2 = \sum_{n=1}^{\infty} (\alpha_{-n}^i \alpha_n^i + n\lambda_{-n}^A \lambda_n^A) + 1,$$

$$A: \quad \frac{1}{2}M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r\lambda_{-r}^A \lambda_r^A - 1,$$

and for closed strings:

$$P: \quad \frac{1}{8}M^2 = \sum_{n=1}^{\infty} (\alpha_{-n}^i \alpha_n^i + n\lambda_{-n}^A \lambda_n^A) + 1 = \sum_{n=1}^{\infty} (\tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + n\tilde{\lambda}_{-n}^A \tilde{\lambda}_n^A) + 1,$$

$$A: \quad \frac{1}{8}M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r\lambda_{-r}^A \lambda_r^A - 1 = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \sum_{r=1/2}^{\infty} r\tilde{\lambda}_{-r}^A \tilde{\lambda}_r^A - 1.$$

These formulae allow us to see that there're tachyonic ground state in the case of A open and closed strings. Usual destination of GSO is to eliminate tachyons and enable supersymmetry. Here we have a bosonic compactified theory, and do not require any SUSY. But according to requirement of the problem we may impose the same GSO projection as in the left-moving part of $SO(32)$ heterotic string. Namely, in A sector of the theory we eliminate all states with an odd number of λ excitations, i.e. we save only positive-valued eigenstates of operator $(-1)^{\sum_r \lambda_{-r}^A \lambda_r^A}$. As a result we will eliminate tachyonic vacuum $|0; k\rangle$ with mass $M^2 = -2$ for open string and $M^2 = -8$ for left- and right-movers of closed string. First excited level after projection is massless. For open string it's spanned by states

$$\alpha_{-1}^i |0; k\rangle, \quad \lambda_{-1/2}^A \lambda_{-1/2}^A |0; k\rangle,$$

for closed string it's spanned by states

$$\alpha_{-1}^i \tilde{\alpha}_{-1}^j |0; k\rangle, \quad \alpha_{-1}^i \tilde{\lambda}_{-1/2}^A \tilde{\lambda}_{-1/2}^B |0; k\rangle, \quad \lambda_{-1/2}^A \lambda_{-1/2}^B \tilde{\alpha}_{-1}^i |0; k\rangle, \quad \lambda_{-1/2}^A \lambda_{-1/2}^B \tilde{\lambda}_{-1/2}^C \tilde{\lambda}_{-1/2}^D |0; k\rangle.$$

In P sector define GSO projection (again, in a manner of that in heterotic string) as $(-1)^F = 1$ with

$$(-1)^F = \bar{\lambda}_0 (-1)^{\sum_n \lambda_{-n}^A \lambda_n^A},$$

where

$$\bar{\lambda}_0 = \lambda_0^1 \lambda_2^0 \cdots \lambda_{32}^0.$$

This projection will reduce by factor two the number of ground states on each level by projecting them onto states with correspondingly fixed chirality. More concrete, in P sector ground state forms reducible representation of $Spin(32)$ algebra, which is projected to irreducible fixed-chirality representation with the help of γ^5-analog $\bar{\lambda}_0$. Therefore vacuum state $|a\rangle$ of open string has 2^{15} states of the mass $M^2 = 2$, and vacuum state $|a\rangle|a\rangle$ of closed string has 2^{30} states of the mass $M^2 = 8$. First excited level has opposite chirality to that of vacuum. For open string it's spanned by $M^2 = 4$ states

$$\alpha^i_{-1}|\dot{a}\rangle, \quad \lambda^A_{-1}|\dot{a}\rangle,$$

for closed string it's spanned by $M^2 = 16$ states

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}|\dot{a}\rangle, \quad \alpha^i_{-1}\tilde{\lambda}^A_{-1}|\dot{a}\rangle, \quad \lambda^A_{-1}\tilde{\alpha}^i_{-1}|\dot{a}\rangle, \quad \lambda^A_{-1}\tilde{\lambda}^B_{-1}|\dot{a}\rangle.$$

Problem 7.3

(i) The dimension of adjoint representation of $SO(n)$ is equal to $\frac{n(n-1)}{2}$, which also gives the number of free fermions, which are transformed in this adjoint representation. Central charge of corresponding energy-momentum tensor is equal to half of the number of fermions or $c = \frac{n(n-1)}{4}$. Substituting this value into the formula

$$c = \frac{k\mathrm{dim}G}{k + \tilde{h}_G}$$

with $\mathrm{dim}G = \mathrm{dim}SO(n) = \frac{n(n-1)}{2}$ and $\tilde{h}_G = n - 2$ one easily gets

$$k = n - 2.$$

(ii) In spinor representation of $SO(16)$ one has 2^{16} fermions of $Spin(16)$, and corresponding central charge is equal to 2^{15}. Therefore

$$k = \frac{14}{240 \cdot 2^{-16} - 1}.$$

Problem 7.4

In the bosonic construction of heterotic string we must first compactify 16 left-movers on T^{16} and then, if we want to, additionally compactify n coordinates from left-movers and n coordinates from right-movers. As a result we will stay with non-compactified $d = 10 - n$ space-time, where there're left-movers and right-movers of bosonic world-sheet fields X^μ, and all of initial right-movers Θ^a, unaffected by compactification, because we compactify bosonic space-time, not superspace Grassmann coordinates Θ^a.

When both left- and right-movers are compactified, which is supposed to be the case in n directions, we may describe geometry of compactifying torus T^n by $\frac{n(n+1)}{2}$-component metric, and accompany this with antisymmetric constant background field with $\frac{n(n-1)}{2}$ components. Together we obtain n^2 degrees of freedom. In addition there're 16 gauge Kaluza-Klein fields, arising from gauge symmetry $U(1)^{16}$, corresponding to translations of compactified 16 additional left-moving coordinates. These Kaluza-Klein gauge fields have their components in n compactified directions of both left- and right-movers. Why only there? Because in flat $d = 10 - n$ directions of uncompactified space-time there're no background fields, from which gauge KK field can originate (as in KK theory: $G_{\mu 25}$: here this metric component is equal to zero, if μ is one of d flat (see for example of bosonic string formula BBS (7.50)) coordinates). Instead, there *are* background G-field components in n directions, which can "attach" to one of 16 purely left-moving gauge degrees of freedom (compact directions). Therefore one has n^2+16n degrees of freedom of background and KK gauge fields, originating from toroidal compactification in bosonic description of heterotic string. This coincides with the number of coordinates of the coset space

$$\mathcal{M}^0_{16+n,n} = \frac{O(16+n,n,R)}{O(16+n,R)\times O(n,R)}.$$

Any lattice of momentums in compactified directions (compact space is also described by values of background fields G, B, A) which in our case has the signature $(16+n, n)$, may be obtained from some given lattice by $O(16+n,n,R)$ group action. We are interested in values of G, B, A background fields, not in values of momentums. Therefore we factorize over momentum characteristic classes: momentums of left-movers and right-movers are left invariant under $O(16+n)$ and $O(n)$ groups independently. Therefore coset space $\mathcal{M}^0_{16+n,n}$ is a space of G, B, A values. It has dimension

$$\dim\mathcal{M}^0_{16+n,n} = \frac{1}{2}\left((16+2n)(15+2n)-(16+n)(15+n)-n(n-1)\right) = n(n+16),$$

which coincides with result obtained above by counting degrees of freedom of G, B, A fields. Finally we have to impose equivalence, originating from T-duality $O(16+n,n,Z)$ gauge group:

$$\mathcal{M}_{16+n,n} = \frac{O(16+n,n,R)}{O(16+n,R)\times O(n,R)\times O(16+n,n,Z)}.$$

Problem 7.5

We are going to start with reconsideration of GSO-type projection rules for different sectors of heterotic string. We will base on fermionic construction of heterotic string with left-moving fermions divided into two groups of equal number of fermions. Denote projector operator as the product of three independent projectors: $(-1)^R \times (-1)^{F_1} \times (-1)^{F_2}$, where $(-1)^R$ stands for the right-moving sector, $(-1)^{F_1}$ is for the first 16 λ^A oscillators, and $(-1)^{F_2}$ is for the next 16 ones. Notice, that we use GSO projection in the right-moving sector because we are going to deal with RNS approach to superstrings, not with GS theory. The reason is due to the formulation of the problem - we are to built non-supersymmetric theory, but GS superstring is initially constructed as supersymmetric.

Remember, that for NS sector an original GSO was to keep negative $(-1)^{2N}$ states (where N is number operator for b-oscillators), and for R sector it was keeping states with positive c-chirality for even number of oscillator excitations, and negative s-chirality for odd number of excitations. Reversing GSO condition for right-moving R-states means reversing of chirality of that state. Let's make the following changes in some of the sectors:

$$(R, A, A), \quad (NS, P, P) \quad \text{change } (-1)^{F_1}\times(-1)^{F_2}$$

This will transform states with right-moving R states and left-moving antiperiodic fermions to that with odd number of excitations in each of A, A left-moving sectors (originally it was even number of excitations due to heterotic prescription of GSO-projection); states with right-moving NS sector will be transformed into that with opposite chirality of both of fermions in left-moving sectors P, P.

Other changes of GSO rules are (F_R stands for GSO operator for roght-moving sector, either R or NS type):

$$(R,P,A),\quad (NS,A,P)\quad \text{change } (-1)^{F_R}\times(-1)^{F_1},$$
$$(R,A,P),\quad (NS,P,A)\quad \text{change } (-1)^{F_R}\times(-1)^{F_2}.$$

By $\tilde{a}$ we will assume zero-point energy of left-movers, which for four possible combinations AA, PP, AP, PA is equal to $1, -1, 0, 0$ respectively. That assumes the following mass formulae for left-movers and for fermionic and bosonic cases of R right-movers and NS right-movers respectively:

$$\frac{1}{8}M^2 = N_R = \tilde{N} - \tilde{a},$$
$$\frac{1}{8}M^2 = N_{NS} - \frac{1}{2} = \tilde{N} - \tilde{a}.$$

Then it obviously follows, there're no tachyons in spectrum. Indeed, right-movers and left-movers always should have the same mass, which is assumed explicitly in formulae above, and number of states can't be negative, but can be half-integer (for non-projected out antiperiodic NS and A cases) which means, that the only possibility for tachyons is when $\tilde{a} = 1$ (which requires AA left-moving configuration), $N_{NS} = 0$ and correspondingly $\tilde{N} = \frac{1}{2}$. To satisfy all these conditions, one should change GSO projection rules of $E_8 \times E_8$ heterotic string for NS, A, A sectors combination. But this is not done in our consideration. Therefore tachyonic absence remains.

As for massless states, all cases according to mass formulae are possible (treated bellow), except $N_{NS} = \frac{1}{2},\ \tilde{N} = 0$. Indeed, that would assume $\tilde{a} = 0$ and therefore $AP,\ PA$ cases for left-movers. But (NS,P,A) and (NS,A,P) are subject of change, namely NS right-moving sector GSO condition is opposite now to that of $E_8 \times E_8$ heterotic string, which means, that N_{NS} is integer-valued now. Only this change alone obviously makes it impossible to satisfy zero-mass equation, but one can notice, that change of A left-moving sector projection makes $\tilde{N}$ half-integer, which also contradicts to zero-mass equation.

Possible massless states are (R,A,A) with $N_R = 0,\ \tilde{N} = 1$; (R,A,P) and (R,P,A) with $N_R = 0,\ \tilde{N} = 0$ and s chiralities of R and P sector fermions; (NS,A,A) with $N_{NS} = \frac{1}{2},\ \tilde{N} = 1$.

Let's study spinor content for massless spectrum. First, consider (R,A,A) with $N_R = 0,\ \tilde{N} = 1$ with necessary odd number of λ and $\tilde{\lambda}$, which makes it impossible to construct left-moving state with the help of $\tilde{\alpha}^i_{-1}$ operator. The left-moving part is $\lambda^A_{-1/2}\tilde{\lambda}^A_{-1/2}|0\rangle_L$, where tilde above lambda means that this is oscillator from the set $A = 17, \ldots 32$ (which shouldn't be mixed with tilde above N, because the later refers to the whole left-moving sector: the sum of states in both left-moving subsectors). Correspondingly, representation of Lorentz group and $SO(16)\times SO(16)$ gauge group is $(\mathbf{8_c}, \mathbf{16}, \mathbf{16})$. Here $\mathbf{16}$ stands for representation of $SO(16)$ gauge symmetry, under which right-moving sector of heterotic string is singlet, and $\mathbf{8_c}$ stands for Lorentz group representation, under which λ^A excitations are singlets. Such a short notation, which combines in one triplet features of both gauge group and Lorentz group representations, is convenient due to the fact, that it occurs, that among fermionic massless states, right-movers and left movers are either transformed under Lorentz group and behave as gauge group singlets, or vice versa. In the case of bosonic massless states this is not so, and notation is a little more complicated.

There're also spinors from sectors $(R,A,P)\oplus(R,P,A)$ for the case of $N_R = 0,\ \tilde{N} = 0$. They form a Lorentz group representation of the type $(\mathbf{8_s}, \mathbf{1}, \mathbf{128_s}) \oplus (\mathbf{8_s}, \mathbf{128_s}, \mathbf{1})$. Singlet denotation **1** has its origin in the condition $\tilde{N} = 0$, which leads to left-moving A state to be $|0\rangle_L$. Also a vitally important note here is that in the notation $\mathbf{128_s}$ for left-movers we mean spinor representation of gauge group $SO(16)$ (one of the two, while the other left-moving sector is singlet under the second $SO(16)$, as pointed above). It is chiral $Spin(16)$, and of course it's Lorentz singlet. Spinor here - is gauge spinor, not space-time spinor.

Let's turn to bosonic massless states. Now we have to write separate triplets (for Lorentz$\times SO(16)\times SO(16)$ group representation) for right-movers and left-movers. The sole massless bosonic state is (NS, A, A) with $N_{NS} = \frac{1}{2}$, $\tilde{N} = 1$. Here GSO projections are the same, as in the case of $E_8 \times E_8$ heterotic string. Obviously, right-moving states are $(\mathbf{8_v}, \mathbf{1}, \mathbf{1})$, which means being space-time vector under Lorentz transformations and singlet under gauge group transformations. According to GSO, the number of λ^A and $\tilde{\lambda}^A$ oscillators should be even. Possible variants are 0 and 2, where if both are 0, then left-moving state is constructed with the help of $\tilde{\alpha}^i_{-1}$ oscillator. Summarizing all these facts, we will get the following representation content of right-left-movers:

$$(\mathbf{8_v}, \mathbf{1}, \mathbf{1})_{\mathrm{R}} \times \left[(\mathbf{8_v}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{120}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{120})\right]_{\mathrm{L}}. \tag{7.66}$$

Here **120** is dimension of adjoint representation of $SO(16)$, which transforms $\lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle_L$ (or $\lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle_L$) from one of A sectors of left-movers, while the second $SO(16)$ transforms vacuum state of another A left-moving sector in a singlet way. Why representation of $SO(16)$ is namely adjoint? The answer is the standard one: because $\lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle_L$ is the product of two vector representations of $SO(16)$, which is adjoint **120** representation.

Observe, that while second and third terms after square brackets opening in (7.66) give Yang-Mills fields, first terms gives gravity multiplet: $\mathbf{8_v} \times \mathbf{8_v} = \mathbf{28} + \mathbf{35} + \mathbf{1}$ - antisymmetric field, dilaton and graviton, being singlets of gauge group. At the same time we didn't obtain gravitino from fermionic massless states.

`Problem` **7.6**

According to Adler-Bardeen theorem anomalies arise in one-loop diagrams with chiral fermion (for gravitational anomalies) and chiral boson (for Yang-Mills anomalies) going around the loop. Chiral massless fields in $SO(16) \times SO(16)$ heterotic string are Majorana-Weyl fermions, constructed in the previous problem:

$$(\mathbf{8_c}, \mathbf{16}, \mathbf{16}), \quad (\mathbf{8_s}, \mathbf{1}, \mathbf{128_s}), \quad (\mathbf{8_s}, \mathbf{128_s}, \mathbf{1}).$$

As it's noted on the pages 174-175 BBS left-handed and right-handed fermions contribute anomalous forms with opposite signs. Here we have spin-$\frac{1}{2}$ fields $\mathbf{8_c}$, $\mathbf{8_s}$ of opposite chirality, contributing forms $\pm I_{1/2}(R)$ of gravitational anomalies. Therefore total anomaly is proportional to the 12-form part of

$$I_{1/2}(F, R) = I_{1/2}(R)\mathrm{tr}_{16\times16}\cos F - I_{1/2}(R)\mathrm{tr}_{128\times1}\cos F - I_{1/2}(R)\mathrm{tr}_{1\times128}\cos F. \tag{7.67}$$

The subscripts here denote representation of $SO(16) \times SO(16)$, in which matrix of two-forms F (that is matrix-represented generators of gauge group, contracted with gauge fields by index of adjoint representation) is written. Pure gravitational anomaly $I_{1/2}(R)$ is a term of total (7.67), coming from zero-order expansion of $\cos F = 1 - \frac{F^2}{2} + \cdots$. It obviously cancels, because $16 \cdot 16 - 128 - 128 = 0$. Therefore equality of numbers of fermions of both chiralities leads to cancellation of gravitational anomaly. Of course the theory is still chiral (chirality assymetric), because different chirality states belong to different representation of gauge group. Defining

$$\mathrm{Tr}\cos F = \mathrm{tr}_{16\times 16}\cos F - \mathrm{tr}_{128\times 1}\cos F - \mathrm{tr}_{1\times 128}\cos F \tag{7.68}$$

we can represent anomaly form as

$$I_{1/2}(F,R) = I_{1/2}(R)\mathrm{Tr}\cos F.$$

Anomalies are 12-form part of $I_{1/2}(F,R)$. We know that gravitational part has an expansion

$$I_{1/2}(R) = 1 + \frac{1}{48}\mathrm{tr}R^2 + \frac{1}{32\cdot 480}\mathrm{tr}R^4 + \frac{1}{72\cdot 64}(\mathrm{tr}R^2)^2 + \cdots ,$$

which together with expansion of $\cos F$ gives a 12-form anomaly:

$$-\frac{1}{2}\mathrm{Tr}F^2\left(\frac{1}{32\cdot 480}\mathrm{tr}R^4 + \frac{1}{72\cdot 64}(\mathrm{tr}R^2)^2\right) + \frac{1}{24\cdot 48}\mathrm{Tr}F^4\mathrm{tr}R^2 - \frac{1}{15\cdot 48}\mathrm{Tr}F^6.$$

Here and later *tr* without index means trace of matrix in fundamental representation. Using indices 1 and 2 to refer to the first and second $SO(16)$, we will be able to write formulae:

$$\mathrm{tr}_{16\times 16}F^2 = 16\mathrm{tr}F_1^2 + 16\mathrm{tr}F_2^2;$$

$$\mathrm{tr}_{128\times 1}F^2 = 16\mathrm{tr}F_1^2;$$

$$\mathrm{tr}_{1\times 128}F^2 = 16\mathrm{tr}F_2^2.$$

Using this formulae observe that according to (7.68) one has

$$\mathrm{Tr}F^2 = \mathrm{tr}_{16\times 16}F^2 - \mathrm{tr}_{128\times 1}F^2 - \mathrm{tr}_{1\times 128}F^2 = 0,$$

which reduces anomaly form to

$$\frac{1}{24\cdot 48}\mathrm{Tr}F^4 trR^2 - \frac{1}{15\cdot 48}\mathrm{Tr}F^6.$$

According to GSW (13.5.4):

$$\frac{1}{15}\mathrm{Tr}F^6 = \frac{1}{4}\left[(\mathrm{tr}F_1^2)^3 + (\mathrm{tr}F_2^2)^3\right] - (\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2)(\mathrm{tr}F_1^4 + \mathrm{tr}F_2^4);$$

$$\frac{1}{24}\mathrm{Tr}F^4 = \frac{1}{4}\left[(\mathrm{tr}F_1^2)^2 + (\mathrm{tr}F_2^2)^2 - \mathrm{tr}F_1^2\mathrm{tr}F_2^2\right] - \mathrm{tr}F_1^4 - \mathrm{tr}F_2^4.$$

These formulae may be used to rewrite anomaly 12-form as

$$\frac{1}{48}X_8\left(\mathrm{tr}R^2 - \mathrm{tr}F_1^2 - \mathrm{tr}F_2^2\right),$$

where $X_8 = \frac{1}{24}\mathrm{Tr}F^4$.

Problem 7.7

Heterotic string in this problem is studied in the fermionic formulation.

Consider $SO(32)$ heterotic string after GSO-type projection (imposed on its left-moving part). We are going to study first excited state with $M^2 = 8$, which is number operators eigenstate with eigenvalues

$$N_R = 1, \quad N_L(A) = 2, \quad N_L(P) = 0.$$

Right-moving states (constructed in space-time supersymmetric GS formalism) are:

$$\alpha^i_{-1}|j\rangle, \quad S^i_{-1}|j\rangle \quad \text{128 bosons;}$$

$$\alpha^i_{-1}|\dot{a}\rangle, \quad S^i_{-1}|\dot{a}\rangle \quad \text{128 fermions.}$$

Left-moving states are

$$A: \ \tilde{\alpha}^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle, \ \tilde{\alpha}^i_{-2}|0\rangle, \ \lambda^A_{-1/2}\lambda^B_{-1/2}\lambda^C_{-1/2}\lambda^D_{-1/2}|0\rangle, \ \lambda^A_{-1/2}\lambda^B_{-3/2}|0\rangle, \ \tilde{\alpha}^i_{-1}\lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle \quad \text{40996 bosons;}$$

$$P: \ |\dot{a}\rangle \quad 2^{15} = 32768 \text{ fermions.}$$

Note, that the number of $\lambda^A_{-1/2}\lambda^B_{-3/2}|0\rangle$ states is $32{\cdot}32$, because this state is constructed with the help of different rising operators. Because numbers of fermions and bosons in the right-moving sector coincide, the total number of bosonic and fermionic left$\times$right tensor product states coincide too. Total number of bosons and fermions is equal to $(40996+32768)\cdot 256 = 18883584$.

Consider now $E_8 \times E_8$ heterotic string theory with GSO-type projection imposed separately on both parts of left-moving sector. We are going to define λ^A with $A = 17, \ldots, 32$ as $\tilde{\lambda}^A$, while saving denotation for others λth. In the form, e.g. AP, it's assumed that A stands for first 16 λ-modes,and P is for the rest ones. Right-moving sector is exactly the same as in the case of $SO(32)$ heterotic string. Therefore here we are going to study left-moving sector. On the first excitation level number operators are equal to:

$$N_L(AA) = 2, \quad N_L(AP) = N_L(PA) = 1, \quad N_L(PP) = 0.$$

In the case of AP and PA sectors we will denote ground state by $|0;a\rangle$ and $|a;0\rangle$ correspondingly, where actually a index may be dotted, which will depict conjugate (oppposite chirality) irreducible representation of $Spin(16)$. Due to GSO-type projection, ground state spinors are denoted with dotted indices, if it's applied an even number of λ-operators to this ground state to get an excited state. Contrarily, if the number of raising operators is odd, then ground state has opposite chirality, which is denoted by undotted index.

The demonstration of this rule is immediate on AP, PA, PP secotrs:

$$AP: \ \tilde{\alpha}^i_{-1}|0;\dot{a}\rangle, \ \lambda^A_{-1/2}\lambda^B_{-1/2}|0;\dot{a}\rangle, \ \tilde{\lambda}^A_{-1}|0;a\rangle \quad \text{18432 fermions;}$$

$$PA: \ \tilde{\alpha}^i_{-1}|\dot{a};0\rangle, \ \tilde{\lambda}^A_{-1/2}\tilde{\lambda}^B_{-1/2}|\dot{a};0\rangle, \ \lambda^A_{-1}|a;0\rangle \quad \text{18432 fermions;}$$

$$PP: \ |\dot{a}\dot{b}\rangle \quad \text{16384 bosons.}$$

These states should be accompanied with AA bosonic sector states:

$$AA: \ \tilde{\alpha}^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle, \ \tilde{\alpha}^i_{-2}|0\rangle, \ \lambda^A_{-1/2}\lambda^B_{-1/2}\lambda^C_{-1/2}\lambda^D_{-1/2}|0\rangle, \ \lambda^A_{-1/2}\lambda^B_{-3/2}|0\rangle, \ \tilde{\alpha}^i_{-1}\lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle,$$

$$\tilde{\lambda}^A_{-1/2}\tilde{\lambda}^B_{-1/2}\tilde{\lambda}^C_{-1/2}\tilde{\lambda}^D_{-1/2}|0\rangle, \ \tilde{\lambda}^A_{-1/2}\tilde{\lambda}^B_{-3/2}|0\rangle, \ \tilde{\alpha}^i_{-1}\tilde{\lambda}^A_{-1/2}\tilde{\lambda}^B_{-1/2}|0\rangle, \ \tilde{\lambda}^A_{-1/2}\tilde{\lambda}^B_{-1/2}\lambda^C_{-1/2}\lambda^D_{-1/2}|0\rangle, \quad \text{20516 bosons.}$$

Altogether, there're 36900 bosons and 36884 fermions. Total number of states is 73764 - as in the case of $SO(32)$ heterotic string. Note, that there're more fermions, but less bosons, despite the sum of states is the same. And again, these states are to be tensored with supersymmetric right-moving sector.

Problem 7.8

(i) It holds $e_i \cdot e_j = 2\delta_{ij}$, which means, that condition $e_i^\star \cdot e_j = \delta_{ij}$ on basis vectors of dual lattice are satisfied by vectors $e_1^\star = \frac{1}{2}(1,1) = \frac{1}{2}e_1$ and $e_2^\star = \frac{1}{2}(1,-1) = \frac{1}{2}e_2$. Lattice is then not self-dual. Metric tensor is $g_{ij} = \mathrm{diag}\{2,\ 2\}$, therefore lattice is not unimodular: $g = \det ||g_{ij}|| = 4$. For any two vectors $v = me_1 + ne_2$ and $w = pe_1 + qe_2$ one has $v \cdot w = pm + nq$, which is integer, therefore lattice is integral. Moreover, $v^2 = 2m^2 + 2n^2$, which is even, and therefore lattice is even.

For dual lattice one has $g_{ij}^\star = \mathrm{diag}\{\frac{1}{2},\ \frac{1}{2}\}$. Lattice is not unimodular $g = \det ||g_{ij}^\star|| = \frac{1}{4}$. It's not integral: for two vectors $v^\star = me_1^\star + ne_2^\star$ and $w^\star = pe_1^\star + qe_2^\star$ one has $v^\star \cdot w^\star = \frac{1}{2}(pm + qn)$, which is not necessary integer.

(ii) Obviously this one works: $e_1 = \frac{1}{\sqrt{2}}(1,1)$, $e_2 = \frac{1}{\sqrt{2}}(1,-1)$. Indeed, for Lorentzian signature $\eta_{ij} = (-1,1)$ and a couple of vectors $e_1^\star = -e_2$, $e_2^\star = -e_1$ one has $e_i \cdot e_j^\star = \delta_{ij}$, therefore $e_i^\star$ are basis vectors of dual lattice. At the same time they apparently belong to the initial lattice. Therefore lattice is self-dual. Then, metric tensor is given by $g_{ij} = \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$, and therefore $\sqrt{|g|} = 1$, which means that lattice is unimodular. For any two vectors $v = me_1 + ne_2$ and $w = pe_1 + qe_2$ of the lattice one has $v \cdot w = -np - mq$, which is integer. Finally, $v^2 = -2mn$, which is even. Therefore lattice is an integral even.

Problem 7.9

(i) We are going to deal with CFT description of only one compactified coordinate; the rest coordinates may be considered independently in a well known CFT bosonic string formalism. The action

$$S[X] = \frac{1}{\pi}\int_M \partial X \bar{\partial} X d^2 z \tag{7.69}$$

indeed may be used for description of bosonic string on torus M, parametrized by coordinates $z,\ \bar{z}$. It follows from the fact that to define torus we should define modular parameter τ, which allows us to identify points on complex plane as

$$z \sim z+1, \quad z \sim z+\tau.$$

After we have done this, we can define metric tensor on torus in such a way, that a measure of integration will be d^2z. Because of of torus is uniquely given by $\tau = \tau_1 + i\tau_2$, and torus may be made uniform by corresponding conformal transformation, one can uniquely parametrize corresponding constant metric on torus via τ_i:

$$g_{\alpha\beta} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}_{\alpha\beta}$$

This metric tensor has $\sqrt{|g|} = 1$. Then we will indeed get action (7.69) (which looks like that in spherical world-sheet case, or Riemann plane without identifications of points).

In this problem we deal with closed bosonic string, because for open string there would be no sense to define winding numbers. Therefore total-derivative terms in variation of the action will be canceled, due to the fact that equations

$$X(z+1,\bar{z}+1) = X(z,\bar{z}) + 2\pi R W_1,$$

$$X(z+\tau,\bar{z}+\bar{\tau}) = X(z,\bar{z}) + 2\pi R W_2$$

will lead to variations of X and its derivatives to be equal at opposite edges of world-sheet (opposite edges of torus moduli space cell on a complex plane). It loks like this:

$$\delta S = \frac{1}{\pi}\int d^2z\left(\partial\delta X\bar{\partial}X + \partial X\bar{\partial}\delta X\right) =$$

$$= -\frac{2}{\pi}\int d^2z\delta X\partial\bar{\partial}X + \frac{1}{\pi}\left(\delta X\bar{\partial}X + \delta X\partial X\right)|_{z,\bar{z}} =$$

$$= -\frac{2}{\pi}\int d^2z\delta X\partial\bar{\partial}X.$$

Therefore equations of motion are $\partial\bar{\partial}X = 0$. Classical instanton solution of this equations is

$$X_{cl} = \frac{2\pi R}{\tau-\bar{\tau}}\left((W_2 - W_1\bar{\tau})z + (W_1\tau - W_2)\bar{z}\right). \tag{7.70}$$

In difference with Riemann-plane case now we don't have any oscillator terms. The thing is that they would violate identifications on complex plane, while in the case of cylinder world-sheet, equivalent to Riemann plane (or Rieman sphere), we didn't have to identify any points on complex plane.
(ii) Note, that $\int_M d^2z = \int_M d\mathrm{Re}z d\mathrm{Im}z = |\tau|\sin\phi(\tau)$, where $\phi(\tau)$ is a phase of τ. Using solution (7.70) we can easily obtain

$$S[X_{cl}] = C(W_2 - W_1\bar{\tau})(W_2 - W_1\tau) = C|W_2 - W_1\tau|^2,$$

where

$$C = -4\pi R^2\frac{|\tau|\sin\phi(\tau)}{(\tau-\bar{\tau})^2} = \frac{\pi R^2}{|\tau|\sin\phi(\tau)},$$

and we've used $\tau - \bar{\tau} = 2i\mathrm{Im}\tau$, and $\sin\phi(\tau) = \frac{\mathrm{Im}(\tau)}{|\tau|}$.
(iii) We can recast classical partition function

$$Z_{cl} = \sum_{W_1,W_2} e^{-S_{cl}(W_1,W_2)}$$

performing Poisson resummation. The aim of this is to find some duality - some sort of interchange of W_1 and W_2 - which won't affect the partition function.

Introduce symmetric matrix

$$A = \frac{C}{\pi}\begin{bmatrix} |\tau|^2 & -\frac{1}{2}(\tau+\bar{\tau}) \\ -\frac{1}{2}(\tau+\bar{\tau}) & 1 \end{bmatrix}.$$

Because of $C > 0$ and $\det A = \frac{C^2}{4\pi^2}(2|\tau|^2 - \tau^2 - \bar{\tau}^2) > 0$, this matrix A is positive-defined due to Sylvester's criterion. With the help of this matrix we can rewrite partition function in the form

$$Z_{cl} = \sum_{\{M\}} \exp\left(-\pi M^T A M\right),$$

where $M = \binom{W_1}{W_2}$. Then, according to Poisson resummation formula

$$Z_{cl}(A) = \frac{1}{\sqrt{\det A}} Z_{cl}(A^{-1}).$$

The inverse matrix is given by

$$A^{-1} = \frac{\pi}{C \det A} \begin{bmatrix} 1 & \frac{1}{2}(\tau + \bar{\tau}) \\ \frac{1}{2}(\tau + \bar{\tau}) & |\tau|^2 \end{bmatrix}.$$

Using this one finds that

$$Z_{cl}(A^{-1}) = \sum_{W_1, W_2} \exp\left(-\frac{\pi^2}{C \det A}|W_1 + \tau W_2|^2\right).$$

Therefore we have the following identity:

$$\sum_{W_1, W_2} \exp\left(-C|W_2 - W_1\tau|^2\right) = \frac{1}{\sqrt{\det A}} \sum_{W_1, W_2} \exp\left(-\frac{\pi^2}{C \det A}|W_1 + \tau W_2|^2\right),$$

which exhibits a duality $W_1 \to \frac{\pi}{C\sqrt{\det A}} W_2$, $W_2 \to -\frac{\pi}{C\sqrt{\det A}} W_1$ (transformation from l.h.s. to r.h.s. of the last equation) of partition function up to a total multiplier.

Problem 7.10

(i) A set of basis vectors of E_8 lattice is a set of simple roots of E_8:

$$e_1 = (0, 0, 0, 0, 0, -1, 1, 0);$$
$$e_2 = (0, 0, 0, 0, -1, 1, 0, 0);$$
$$e_3 = (0, 0, 0, -1, 1, 0, 0, 0);$$
$$e_4 = (0, 0, -1, 1, 0, 0, 0, 0);$$
$$e_5 = (0, -1, 1, 0, 0, 0, 0, 0);$$
$$e_6 = (-1, 1, 0, 0, 0, 0, 0, 0);$$
$$e_7 = \frac{1}{2}(1, -1, -1, -1, -1, -1, -1, 1);$$
$$e_8 = (1, 1, 0, 0, 0, 0, 0, 0).$$

(ii) Suppose $v = \sum_i v^i e_i$ and $w = \sum_j w^j e_j$, where $v^i,\ w^j \in Z$, are two arbitrary vectors from E_8 lattice. Then

$$v \cdot w = \sum_{i,j} v^i w^j e_i \cdot e_j$$

is obviously an integer value, because $e_i \cdot e_j = \{0,\ -1,\ 2\}$. It also holds

$$v^2 = 2\sum_{i=1}^{8}(v^i)^2 - 2\sum_{i=1}^{6} v^i v^{i+1} - 2v^5 v^8,$$

which is even. Therefore E_8 lattice is even.

Denoting by u_i a 8D vector with all zero entries except unit at the ith place, one can expand any 8D vector v as $v = \sum v_i u_i$. It would be a vector of dual lattice $\Gamma_8^\star$ if and only if $v \cdot e \in Z$ for all vectors $e \in \Gamma_8$ - E_8 root lattice. Consider first Γ_8 vectors of the type $e = \pm u_i \pm u_j$ (fix for concreteness some $i \neq j$). Then $v.e = \pm v_i \pm v_j$. This should be an integer. Because of arbitrariness of choice of $i,\ j$, one concludes that all v_i at the same time should be either integer or half-integer. Second, consider vector $e_0 = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$, which belongs to lattice Γ_8. Then $v \cdot e_0 = \frac{1}{2}\sum v_i$, which requires $\sum v_i$ to be even. If all v_i are integers, whose sum is even, then it belongs to Γ_8, namely it's an integral combination of vectors $\pm u_i \pm u_j$ of Γ_8. If all v_i are half-integers, whose sum is even, then $v - e_0$ has all integer entries, whose sum is even too. Therefore $v - e_0 \in \Gamma_8$ and thus $v \in \Gamma_8$. As a result one concludes that $\Gamma_8^\star$, which is composed of all vectors whose inner product with Γ_8 vectors is integer, actually is included inside Γ_8. From the other point of view, inner product of any pair of Γ_8 vectors is obviously integer (see Cartan matrix), which means that Γ_8 is sublattice of $\Gamma_8^\star$. Therefore one concludes, that $\Gamma_8 = \Gamma_8^\star$.

`Problem` **7.11**

Consider Γ_{16} lattice with basis vectors $s = \frac{1}{2}(u_1 - u_2 + u_3 - u_4 + \cdots + u_{15} - u_{16})$, $e_i = u_i - u_{i+1},\ i = 2, \ldots 15$, $e_{16} = u_{15} + u_{16}$, where as in the solution of Problem 7.10 u_i stands for vector with unit value at ith place and all zero values at other places. Considerations of absolutely the same scheme as used in the solution of that problem may be employed here to prove that Γ_{16} is an even self-dual lattice. Since $e_1 = u_1 - u_2$ is obviously an integer linear combination of introduced vectors, then Γ_{16} contains $SO(32)$ roots $\pm u_i \pm u_j,\ i \neq j$. In addition it contains one of $Spin(32)$ weight s. The group $Spin(32)$ contains a center $Z_2 \times Z_2$. Therefore, Γ_{16} is a weight lattice of $Spin(32)/Z_2$.

`Problem` **7.12**

1. Torus metric properties are characterized by symmetrical metric tensor G. At the same time torus is uniquely defined by modular parameter τ. Therefore metric tensor and modular parameter could be expressed through each other. But τ is not unique for a given torus: it may be transformed to equivalent values by modular transformation group $SL(2, Z)$. Therefore, if one changes τ, one should also reparametrize world-sheet (torus coordinates) to have interval on world-sheet invariant. And if after that one makes a reverse reparametrization, everything returns to the starting values. To summarize: the action of modular $SL(2, Z)$ transformation ought to affect coordinates of world-sheet at the same time with world-sheet metric in a way leaving interval on world-sheet invariant; then one makes a backward reparametrization, which returns both metric and coordinates on world-sheet to their initial values. This contemplation proves the fact, that τ modular transformations $\tau \to \tau + 1,\ \tau \to -\frac{1}{\tau}$ leaves the spectrum of bosonic string invariant.

2. The following transformations (where $\mathcal{G}$ is given by BBS (7.70))

$$\mathcal{G} \to A\mathcal{G}A^T, \quad \begin{pmatrix} W \\ K \end{pmatrix} \to A \begin{pmatrix} W \\ K \end{pmatrix},$$

where $A = \begin{pmatrix} 1_n & 0 \\ N & 1_n \end{pmatrix}$ with integer-valued antisymmetric N matrix, are shift T-duality $O(n,n,Z)$ transformations of bosonic string. As for antisymmetric field B, it's transformed under these transformation as $B_{IJ} \to B_{IJ} + \frac{1}{2}N_{IJ}$. In the case of T^2 it's simply $B_{12} \to B_{12} + \frac{1}{2}N_{12}$. Therefore, as partial case, $\rho \to \rho + 1$, which is $B_{12} \to B_{12} + 1$ (see solution to Exercise 7.8), is retrieved with $N_{12} = 2$.

3. To deal with other T-dualities it's most convenient to employ reparametrizartion invariance of world-sheet torus, which allows us to eliminate 2 degrees of freedom, which we choose to be $G_{12}(= G_{21})$ and B_{12}. Therefore we are left with $\tau = i\frac{\sqrt{\det G}}{G_{22}}$ and $\rho = i\sqrt{\det G}$ (again, see solution to Exercise 7.8). Then, if the transformation of interest is $\tau \to \tau$, $\rho \to -\frac{1}{\rho}$, it leaves $\tau = i\tau_2 = i\frac{\sqrt{\det G}}{G_{22}}$ invariant and transforms $\rho = i\rho_2 = i\sqrt{\det G} \to \frac{i}{\sqrt{\det G}}$. These transformations obviously assume

$$G_{IJ} = \begin{pmatrix} G_{11} & 0 \\ 0 & G_{22} \end{pmatrix}_{IJ} \to \begin{pmatrix} \frac{1}{G_{22}} & 0 \\ 0 & \frac{1}{G_{11}} \end{pmatrix}_{IJ}.$$

Before proceeding, let's interchange toroidally compactified coordinates: $X^{24} \leftrightarrow X^{25}$. This will not change a spectrum, but we will believe, that we've interchanged winding and KK excitation numbers in both of compactified directions. Therefore we got metric of torus:

$$G'_{IJ} = \begin{pmatrix} \frac{1}{G_{11}} & 0 \\ 0 & \frac{1}{G_{22}} \end{pmatrix}_{IJ}.$$

In BBS T-duality for toroidally compactified string is studied in terms of matrix

$$\mathcal{G}^{-1} = \begin{pmatrix} 2(G - BG^{-1}B) & BG^{-1} \\ -G^{-1}B & \frac{1}{2}G^{-1} \end{pmatrix},$$

which after reparametrization of torus made above reduces to

$$\mathcal{G}^{-1} = \begin{pmatrix} 2G & 0 \\ 0 & \frac{1}{2}G^{-1} \end{pmatrix}.$$

Transformation studied here (together with coordinate interchange) brings it to

$$\mathcal{G}'^{-1} = \begin{pmatrix} 2G^{-1} & 0 \\ 0 & \frac{1}{2}G \end{pmatrix}.$$

That is a dual matrix, entering mass condition (BBS (7.68))

$$\frac{1}{2}M_0^2 = (W,\ K)\,\mathcal{G}^{-1}\begin{pmatrix} W \\ K \end{pmatrix},$$

will not change this condition, if one interchanges simultaneously $W^I \to W'^I = \frac{1}{2}K_I$, and $K_I \to K'_I = 2W^I$. This change is also good due to the fact, that it will not affect $N_R - N_L = W^I K_I$. That proves the duality character of considered transformation.

4. Consider the action of transformation $U:\ (\rho,\tau) \to (\tau,\rho)$. Again, we assume reparametrization of torus was made, as in the previous point. Then U-transformation obviously means

$$G_{IJ} = \begin{pmatrix} G_{11} & 0 \\ 0 & G_{22} \end{pmatrix}_{IJ} \to \begin{pmatrix} G_{11} & 0 \\ 0 & \frac{1}{G_{22}} \end{pmatrix}_{IJ}.$$

This transformation resembles partly the case of previous point with only one difference: now T-duality is made along only X^{25} coordinate. The rest is the same, just leave winding and KK excitation numbers unchanged for X^{24} direction and change that for X^{25} direction as described in the previous point.
5. Obviously for reparametrized torus transformations$(\tau,\rho) \to (-\bar{\tau},-\bar{\rho})$ will simply leave τ and ρ unchanged, because for reparametrized in a described in point 3 manner torus τ and ρ are purely imaginary.

`Problem` **7.13**
(i) We will use formulae for background fields T-duality transformations, derived in the solution of Problem 6.5. We start with rectangular metric on T^3: $G_{xx} = G_{yy} = G_{zz} = R^2$ and antisymmetric 2-form with only non-zero component being $B_{xy} = -B_{yx} = Nz$. Other components are zero. Then we perform a T-duality in the x direction, which gives us the following non-zero components:

$$\tilde{G}_{xx} = \frac{1}{R^2},\ \tilde{G}_{xy} = \frac{Nz}{R^2},\ \tilde{G}_{yy} = R^2 + \frac{N^2z^2}{R^2},\ \tilde{G}_{zz} = R^2.$$

All components of $\tilde{B}_{IJ}$ are zero. Now we perform a T-duality in y direction. This gives us

$$\hat{G}_{xx} = \frac{1}{R^2},\ \hat{G}_{yy} = \frac{R^2}{R^4+N^2z^2},\ \hat{G}_{zz} = R^2, \quad \hat{B}_{yx} = -\hat{B}_{xy} = \frac{Nz}{R^4+N^2z^2}$$

with other components being zero.
(ii) Evidently, translation of z coordinate by period of circle which it parametrizes will not change metric components and will affect B-field in a way compensated by shift duality symmetry BBS (7.72). Also as one can easily see situation is more complicated in the theory resulted after two T-duality transformations.
(iii) As soon as translation of z by its period is to be symmetry of the theory, then one concludes that this symmetry is not associated with just geometric T-duality of torus. Then this is nongeometric duality.

8 M-theory and string duality

`Problem` **8.1**
We are going to deal with bosonic part of $D = 11$ supergravity, given by BBS (8.8):

$$S_b = \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{12\kappa_{11}^2}\int A_3\wedge F_4\wedge F_4,$$

where $F_4 = dA_3$. Before starting variation of action let's refine some points. The last term in the action may be rewritten as

$$-\frac{1}{12\kappa_{11}^2}\int A_3\wedge F_4\wedge F_4 =$$

$$= -\frac{1}{12\kappa_{11}^2}\frac{3!4!4!}{11!}\int A_{MNP}(\partial_Q A_{ZRS})(\partial_V A_{XYW})dx^M\wedge dx^N\wedge dx^P\wedge dx^Q\wedge dx^Z\wedge dx^R\wedge dx^S\wedge dx^V\wedge dx^Y\wedge dx^W =$$

$$= -\frac{1}{12\kappa_{11}^2}\frac{1}{11550}\int d^{11}x\varepsilon^{MNPQZRSVXYW}A_{MNP}(\partial_Q A_{ZRS})(\partial_V A_{XYW}) =$$

$$= -\frac{1}{12\kappa_{11}^2}\frac{1}{11550}\int d^{11}x\sqrt{-G}E^{MNPQZRSVXYW}A_{MNP}(\partial_Q A_{ZRS})(\partial_V A_{XYW}).$$

Here

$$E^{MNPQZRSVXYW} = \frac{\varepsilon^{MNPQZRSVXYW}}{\sqrt{-G}} \tag{8.71}$$

is a tensor value. Similarly

$$E_{MNPQZRSVXYW} = \sqrt{-G}\varepsilon_{MNPQZRSVXYW}$$

is a tensor value too, which is useful for the construction of a volume form. We will aply it in the solution of Problem 8.2 for AdS_4.

By definition

$$|F_4|^2 = \frac{1}{4!}G^{M_1N_1}G^{M_2N_2}G^{M_3N_3}G^{M_4N_4}F_{M_1M_2M_3M_4}F_{N_1N_2N_3N_4}.$$

From $F_4 = dA_3$ and

$$A_3 = \frac{1}{3!}A_{ZRS}dx^Z \wedge dx^R \wedge dx^S$$

it follows, that

$$F_4 = \frac{1}{4!}F_{QZRS}dx^Q \wedge dx^Z \wedge dx^R \wedge dx^S,$$

with

$$F_{QZRS} = \frac{4!}{3!}\partial_{[Q}A_{ZRS]} = 4\partial_{[Q}A_{ZRS]}.$$

Note also, that

$$\delta\sqrt{-G} = -\frac{1}{2}\sqrt{-G}G_{MN}\delta G^{MN},$$

$$\delta\int d^{11}x\sqrt{-G}R = \int d^{11}x\sqrt{-G}\left(R_{MN} - \frac{1}{2}RG_{MN}\right)\delta G^{MN}.$$

Now we can proceed to finding equations of motion. First, let's vary the action S_b by G^{MN}. The dependence on metric G^{MN} is a feature of the following term of S_b:

$$\frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right).$$

Variation by G^{MN} gives equations of motion

$$R_{MN} = \frac{1}{2}\left(R - \frac{1}{2}|F_4|^2\right)G_{MN} + \frac{1}{12}F_{MPQR}F_N^{\ PQR}. \tag{8.72}$$

The dependence of the action S_b on 3-form field A_3 is shown in terms

$$-\frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-G}\frac{1}{2}|F_4|^2 - \frac{1}{12\kappa_{11}^2}\int A_3 \wedge F_4 \wedge F_4,$$

which according to calculations above may be represented in the form

$$-\frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-G}\frac{1}{2}|F_4|^2-\frac{1}{12\kappa_{11}^2}\frac{1}{11550}\int d^{11}x\varepsilon^{MNPQZRSVXYW}A_{MNP}(\partial_Q A_{ZRS})(\partial_V A_{XYW}).$$

Lagrange equations of motion

$$\partial_Q\frac{\partial L}{\partial(\partial_Q A_{ZRS})}-\frac{\partial L}{\partial A_{ZRS}}=0$$

give us

$$-\frac{1}{4}\partial_Q\left(\sqrt{-G}F^{QZRS}\right)-\frac{1}{12}2\varepsilon^{MNPQZRSVXYW}\frac{1}{11550}\partial_Q\left(A_{MNP}\partial_V A_{XYW}\right)-$$

$$-\frac{1}{12}\varepsilon^{MNPQZRSVXYW}\frac{1}{11550}(\partial_Q A_{MNP})(\partial_V A_{XYW})=0.$$

Using (8.71) this equation of motion may be recast in the form

$$\partial_Q\left(\sqrt{-G}\left(3F^{QZRS}+\frac{2}{11550}E^{MNPQZRSVXYW}A_{MNP}\partial_V A_{XYW}\right)\right)+$$

$$+\frac{1}{11550}\varepsilon^{MNPQZRSVXYW}(\partial_Q A_{MNP})(\partial_V A_{XYW})=0.$$

This equation may be also rewritten as

$$\partial_Q\left(\sqrt{-G}\left(3F^{QZRS}+\frac{2}{11550}E^{MNPQZRSVXYW}A_{MNP}F_{VXYW}\right)\right)+$$

$$+\frac{1}{11550}\varepsilon^{MNPQZRSVXYW}F_{QMNP}F_{VXYW}=0. \tag{8.73}$$

`Problem` **8.2**

Define the strength of A_3 on the first four coordinates in AdS_4 (labeled by Greek indices) as

$$F_4=M\varepsilon_4,$$

and zero on the other seven coordinates S^7 (labeled by Latin indices). Here ε_4 is volume form on $d=4$ space-time:

$$\varepsilon_4=\frac{1}{4!}E_{\mu\nu\lambda\rho}dx^\mu\wedge dx^\nu\wedge dx^\lambda\wedge dx^\rho,$$

where as was mentioned in the solution of the Problem 8.1 $E_{\mu\nu\lambda\rho}=\sqrt{-g_{(4)}}\varepsilon_{\mu\nu\lambda\rho}$ is a tensor value. Here $g_{(4)}$ is determinant of restriction of metric tensor on AdS_4. Then

$$F_{\mu\nu\lambda\rho}=M\sqrt{-g_{(4)}}\varepsilon_{\mu\nu\lambda\rho},$$

and

$$F^{\mu\nu\lambda\rho}=M\frac{1}{\sqrt{-g_{(4)}}}\varepsilon^{\mu\nu\lambda\rho}. \tag{8.74}$$

Therefore

$$|F_4|^2 = \frac{M^2}{24}\varepsilon^{\mu\nu\lambda\rho}\varepsilon_{\mu\nu\lambda\rho} = -M^2,$$

where we've used the fact that $\varepsilon^{0123} = \eta^{00}\eta^{11}\eta^{22}\eta^{33}\varepsilon_{0123} = -1$. Observe, that 3-form field A_3 has its components only on AdS_4, which means that second set of supergravity bosonic equations of motion (8.73) is reduced to

$$\partial_\theta \left(\sqrt{-G}\left(3F^{\theta\zeta\rho\sigma} + \frac{2}{11550}E^{\mu\nu\pi\theta\zeta\rho\sigma\tau\chi\upsilon\omega}A_{\mu\nu\pi}F_{\tau\chi\upsilon\omega}\right)\right) +$$

$$+\frac{1}{11550}\varepsilon^{\mu\nu\pi\theta\zeta\rho\sigma\tau\chi\upsilon\omega}F_{\theta\mu\nu\pi}F_{\tau\chi\upsilon\omega} = 0. \tag{8.75}$$

Note, that the last two terms are obviously zero, because they are trying to antisymmetrize eleven four-valued indices indices in eleven-component ε-symbol. Due to the fact that our space-time has topology $AdS_4 \times S^7$, we have $G = g_{(4)}g_{(7)}$. Taking this and formula (8.74) into account, one goes from the first term of (8.75) to

$$\partial_\mu \left(\sqrt{g_{(7)}}\varepsilon^{\mu\nu\lambda\rho}\right) = 0.$$

This equation is indeed satisfied, because S^7 metric determinant $g_{(7)}$ doesn't depend on AdS_4 coordinates with Greek indices, and $\varepsilon^{\mu\nu\lambda\rho}$ is just constant.

Now let's proceed to the first set of $D = 11$ supergravity equations (8.72). The Ricci tensor of proposed solution is

$$R_{\mu\nu} = -(M_4)^2 g_{\mu\nu}, \qquad R_{ij} = (M_7)^2 g_{ij},$$

and our aim is to show that it satisfies (8.72). First note that $R = g^{\mu\nu}R_{\mu\nu} + g^{ij}R_{ij} = -4M_4^2 + 7M_7^2$, where it's taken into account, that $g_{\mu\nu}g^{\mu\nu} = 4$. Then, for AdS_4 with respect to (8.75) and the fact that $E_{\mu\sigma\lambda\rho}E_\nu{}^{\sigma\lambda\rho} = -6g_{\mu\nu}$, equations (8.72)

$$R_{\mu\nu} = \frac{1}{2}\left(R - \frac{1}{2}|F_4|^2\right)g_{\mu\nu} + \frac{1}{12}F_{\mu\sigma\lambda\rho}F_\nu{}^{\sigma\lambda\rho}$$

mean that

$$M^2 = 14M_7^2 - 4M_4^2.$$

For S^7 equations (8.72) are written as

$$R_{ij} = \frac{1}{2}Rg_{ij} - \frac{1}{4}|F_4|^2 g_{ij},$$

which gives

$$M^2 = 8M_4^2 - 10M_7^2.$$

Then we conclude that $M_4^2 = 2M_7^2$ and $M^2 = 6M_7^2$.

Problem 8.3

According to Hawking, Ellis [7], conformal transformation $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ affects scalar curvature $R \to \tilde{R}$ in the following manner:

$$R = \Omega^2\tilde{R} + 2(n-1)\Omega^{-1}\Omega_{;\mu}^{;\mu} + (n-1)(n-4)\Omega^{-2}g^{\mu\nu}\Omega_{,\mu}\Omega_{,\nu}.$$

Dimension of space-time is $n = 10$, which we will substitute later. Here we go from string metric $g_{\mu\nu}$ to Einstein metric $g^E_{\mu\nu}$. Then we replace notation $g^E_{\mu\nu}$ by $g_{\mu\nu}$. Because of $g_{\mu\nu} = e^{\frac{\Phi}{2}} g^E_{\mu\nu}$, which in the notation above means $\hat{g}_{\mu\nu} = e^{-\frac{\Phi}{2}} g_{\mu\nu}$, therefore $\Omega = e^{-\frac{\Phi}{4}}$. Relation between determinants of metrics is $\hat{g} = e^{-5\Phi} g$, inverse metrics: $\hat{g}^{\mu\nu} = e^{\frac{\Phi}{2}} g^{\mu\nu}$. D'Alambertian is equal to:

$$\Omega^{;\mu}_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} \Omega^{,\mu} \right).$$

Therefore

$$\int d^{10}x \sqrt{-g} e^{-2\Phi} R =$$

$$= \int d^{10}x \sqrt{-\hat{g}} \tilde{R} - \frac{n-1}{2} \int d^{10}x e^{-\frac{7\Phi}{4}} \left(\hat{g}^{\mu\nu} e^{\frac{7\Phi}{4}} \sqrt{-\hat{g}} \Phi_{,\nu} \right)_{,\mu} + \int d^{10}x \frac{(n-1)(n-4)}{16} \Phi_{,\mu} \Phi_{,\nu} \hat{g}^{\mu\nu}.$$

Partial integration of the second term and rising of indices in Einstein frame will give us

$$\int d^{10}x \sqrt{-\hat{g}} \left[\tilde{R} + \frac{(n-1)(n-18)}{16} \Phi_{,\mu} \Phi^{,\mu} \right],$$

which for $n = 10$ finally means

$$\int d^{10}x \sqrt{-\hat{g}} \left[\tilde{R} - \frac{9}{2} \Phi_{,\mu} \Phi^{,\mu} \right].$$

At the same time

$$\int d^{10}x \sqrt{-g} e^{-2\Phi} \Phi_{,\mu} \Phi_{,\nu} g^{\mu\nu} = \int d^{10}x \sqrt{-\hat{g}} \Phi_{,\mu} \Phi^{,\mu},$$

where on the r.h.s. indices are risen with the help of new - Einstein-frame - metric tensor. Combining obtained results one will get S_{NS} action:

$$S_{NS} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} e^{-\Phi} |H_3|^2 \right).$$

Here we have used the fact that according to BBS (8.11) it takes place

$$|H_3|^2 = \frac{1}{3!} g^{M_1 N_1} g^{M_2 N_2} g^{M_3 N_3} H_{M_1 M_2 M_3} H_{N_1 N_2 N_3} \to e^{-\frac{3\Phi}{2}} |H_3|^2.$$

Squared field strengths' $|F_2|^2$, $|\tilde{F}_4|^2$ are transformed in a similar way, which should be used in in the transition to S_R in Einstein frame.

Problem 8.4

Our aim is to redefine forms C_1 and C_3, which are dynamical fields in the actions BBS (8.41), (8.42), to get factor $e^{-2\Phi}$ in that actions. For this aim define $C_1 = e^{-\Phi} \tilde{C}_1$ and $C_3 = e^{-\Phi} \tilde{C}_3$. Then, e.g.:

$$F_2 = dC_1 = d(e^{-\Phi} \tilde{C}_1) = e^{-\Phi} (d\tilde{C}_1 - d\Phi \wedge \tilde{C}_1) = e^{-\Phi} \tilde{F}_2,$$

where $\tilde{F}_2 = d\tilde{C}_1 - d\Phi \wedge \tilde{C}_1$. Squaring F_2 will lead to factor $e^{-2\Phi}$ in the corresponding term. Redefinition of F_4 works in exactly the same manner.

Problem 8.5
Consider Chern-Simons action

$$S_{CS} = -\frac{1}{4\kappa^2}\int B_2 \wedge F_4 \wedge F_4.$$

$U(1)$ gauge transformation of 1-form $\delta A_1 = d\Lambda$ assumes also transformation of the 3-form: $\delta A_3 = d\Lambda \wedge B$, where $B = B_2$ is a 2-form (of supergravity multiplet). Then for $F_4 = dA_3$ one obtains gauge transformation law: $\delta F_4 = d\Lambda \wedge dB_2$ (non-zero in difference with transformation law BBS (8.33) for $\tilde{F}_4$)

$$\delta S_{CS} = -\frac{1}{2\kappa^2}\int B_2 \wedge F_4 \wedge \delta F_4 = -\frac{1}{2\kappa^2}\int B_2 \wedge F_4 \wedge d\Lambda \wedge dB_2.$$

Because of forms $d\Lambda,\ F_4$ are closed, one can conclude that:

$$\delta S_{CS} = -\frac{1}{4\kappa^2}\int d\left(B_2 \wedge F_4 \wedge d\Lambda \wedge B_2\right).$$

Therefore $U(1)$ variation of the action S_{CS} is given by a surface integral, which means $U(1)$ invariance of S_{CS}.

Problem 8.6
First, we are going to study transformation of the action

$$S_{NS} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right)$$

According to the statement of the present problem, we replace $\Phi \to -\Phi$. Then we get

$$S_{NS} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right)$$

Using the same technics as in the solution of Problem 8.3, for n being a dimension of space-time, one finds that for $\hat{g}_{\mu\nu} = e^{\Phi}g_{\mu\nu}$ it holds

$$\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{2\Phi}R = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\hat{g}}e^{-2\Phi}\left(\tilde{R} + \frac{(n-1)(n-10)}{4}\partial_\mu\Phi\partial^\mu\Phi\right).$$

In parallel one has

$$\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{2\Phi}4\partial_\mu\Phi\partial^\mu\Phi = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\hat{g}}e^{-2\Phi}4\partial_\mu\Phi\partial^\mu\Phi,$$

where as always on the r.h.s. summation $\partial_\mu\Phi\partial^\mu\Phi$ is performed with the help of new metric $\hat{g}_{\mu\nu}$ (we actually make a replacement of metric, therefore in final formulae hats from all g may be deleted). One also has

$$\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{2\Phi}\left(-\frac{1}{2}|H_3|^2\right) = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\hat{g}}\left(-\frac{1}{2}|H_3|^2\right),$$

on the r.h.s. the value $|H_3|^2$ is constructed with the help of new metric. Therefore transformed NS part of the action of type-IIB supergravity looks like

$$S_{NS} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\hat{g}}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - e^{2\Phi}\frac{1}{2}|H_3|^2\right)$$

Second, we consider transformation of R part of supergravity action:

$$S_R = -\frac{1}{4\kappa^2}\int d^{10}x\sqrt{-g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right), \tag{8.76}$$

where $F_n = dC_n$. As it's recommended in the statement of the problem, we begin our consideration with the case of $C_0 = 0$, and therefore $F_1 = 0$, $\tilde{F}_3 = F_3 - C_0 \wedge H_3 = F_3$. At the same by definition $\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$. Taking this into account, and performing transition to the new metric (initially there's no dilaton dependence in S_R, which omits the first step $\Phi \to -\Phi$ of transformation), one results in the action

$$S_R = -\frac{1}{4\kappa^2}\int d^{10}x\sqrt{-\hat{g}}\left(e^{-2\Phi}|F_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right),$$

Third, Chern-Simons action

$$S_{CS} = -\frac{1}{4\kappa^2}\int C_4 \wedge H_3 \wedge F_3,$$

remains unchanged under proposed transitions, because it doesn't contain any dilaton dependence and as topological term it doesn't depend on metric.

Now the task is to figure out which theory we resulted in. According to field content, it must be again type-IIB supergravity. To verify that it indeed is, observe that as in the solution of Problem 8.4 we can use redefinitions of C_n and B_2 fields and their strengths'. This is not change of physical meaning, just change of notation in the same fields multiplet, made independently on the origin of the fields (remember, that B and C fields have different origin: as $R-R$ gauge fields and as reduction of $D = 11$ 3-form A_3 respectively) locally in considered $D = 10$ supergravity. Then let's redefine $B_2 \to e^{-\Phi}B_2$, and correspondingly $H_3 \to e^{-\Phi}H_3'$, where $H_3' = dB_2 - d\Phi\wedge B_2$. Also replace $C_2 \to e^{\Phi}C_2$, and respectively $F_3 \to e^{\Phi}F_3'$, where $F_3' = dC_2 + d\Phi \wedge C_2$. Note, that then Chern-Simons term of bosonic part of type-IIB supergravity will take the proper form

$$S_{CS} = -\frac{1}{4\kappa^2}\int C_4 \wedge H_3' \wedge F_3',$$

without any need of redefinition of C_4. And that is good, because otherwise the term with $|\tilde{F}_5|^2$ in S_R would be irreversibly damaged.

We also have

$$S_{NS} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\hat{g}}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3'|^2\right),$$

$$S_R = -\frac{1}{4\kappa^2}\int d^{10}x\sqrt{-\hat{g}}\left(|F_3'|^2 + \frac{1}{2}|\tilde{F}_5|^2\right),$$

which is exactly the form of actions we need to conclude, that we are dealing with type-IIB supergravity.

Therefore the transition made in this problem describes duality type-IIB-type-IIB. This is the duality between theories with inverse coupling constant, which is the consequence of the fact, that string coupling constant g_s has the form of exponential function of dilaton vev.

This duality may be generalized for the case of $C_0 \neq 0$. We must reformulate some steps of the previous consideration. Action S_R has now the most general view (8.76). Redefinition of metric transforms it to

$$S_R = -\frac{1}{4\kappa^2} \int d^{10}x \sqrt{-\hat{g}} \left(e^{-4\Phi}|F_1|^2 + e^{-2\Phi}|\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2 \right).$$

To return to the form of type-IIB supergravity action, one should perform a change of variables $C_0 \to e^{2\Phi} C_0$, correspondingly to which field strength is changed to $F_1 \to e^{2\Phi} F_1'$, where $F_1' = dC_0 + 2d\Phi \wedge C_0$. Considered above transformation $C_2 \to e^{\Phi} C_2$, $F_3 \to e^{\Phi} F_3'$ is more subtle now. Indeed, now we are dealing with $\tilde{F}_3 = F_3 - C_0 \wedge H_3 = F_3$, not just $F_3 = dC_2$. As before one should perform a transformation $B_2 \to e^{-\Phi} B_2$, and correspondingly $H_3 \to e^{-\Phi} H_3'$, which relates to S_{NS} action, unaffected by any value of C_0. Then observe, that if one still makes $C_2 \to e^{\Phi} C_2$, $F_3 \to e^{\Phi} F_3'$, then, because of $C_0 \wedge H_3 \to e^{2\Phi} C_0 \wedge e^{-\Phi} H_3' = e^{\Phi} C_0 \wedge H_3'$, one has $\tilde{F}_3 \to e^{\Phi} \tilde{F}_3'$. Then we arrive to the action

$$S_R = -\frac{1}{4\kappa^2} \int d^{10}x \sqrt{-\hat{g}} \left(|F_1'|^2 + |\tilde{F}_3'|^2 + \frac{1}{2}|\tilde{F}_5|^2 \right)$$

of the required type-IIB supergravity form.

Problem 8.7

With the help of results considering transformation of S_{NS} from the solution of Problem 8.6, the solution of this problem is obvious. Let's point out some additional things. Considered transformation 1) $\Phi \to -\Phi$, 2)$g_{\mu\nu} \to e^{-\Phi} g_{\mu\nu}$ coincides with reverse to it. Indeed, repeating of this transformation brings everything back: 1) $e^{-\Phi} g_{\mu\nu} \to e^{\Phi} g_{\mu\nu}$, $-\Phi \to \Phi$, 2) $e^{\Phi} g_{\mu\nu} \to e^{-\Phi} e^{\Phi} g_{\mu\nu} = g_{\mu\nu}$.

Remember also how according to the end of the solution of Problem 8.3 terms of the type $|F_n|^2$ are transformed under change of metric. One should also take into account, that, as was shown in the solution of Problem 5.12, $\mathrm{tr}(F^2) = \frac{1}{30}\mathrm{Tr}(F^2)$, if F is generator of $SO(32)$. Then actions BBS (8.73) and BBS (8.81) of type-I and $Spin(32)/Z_2$ heterotic supergravities are mapped onto each other by considered two-step duality transformation. Correspondence between 2-form fields is $B_2 \leftrightarrow C_2$, which is equivalent to $\tilde{H}_3 \leftrightarrow \tilde{F}_3$.

Problem 8.8

Conformal transformation $g_{\mu\nu} \to e^{-\Phi} g_{\mu\nu}$ may be recast in terms of zehnbein transformation: $e^a_\mu \to e^{-\Phi/2} e^a_\mu$. This also means, that $e^\mu_a \to e^{\Phi/2} e^\mu_a$. Therefore Dirac matrices in curved space-time $\Gamma^\mu = e^\mu_a \Gamma^a$ transform as follows: $\Gamma^\mu \to e^{\Phi/2}\Gamma^\mu$, and $\Gamma_\mu \to e^{-\Phi/2}\Gamma_\mu$. Spin connection may be expressed through metric (see GSW (12.1.5), where $\nabla g = 0$ is assumed). With the help of this formula one finds transformation law for spin connection under conformal transformation of metric:

$$\omega^{ab}_\mu \to \omega^{ab}_\mu + \frac{1}{4}\left(e^{\rho a} e^b_\mu - e^a_\mu e^{\rho b} \right) \Phi_{,\rho}.$$

This will lead to transformation of covariant derivative BBS (8.18) of spinors:

$$\nabla_\mu\varepsilon \to \nabla'_\mu\varepsilon = \nabla_\mu\varepsilon + \frac{1}{8}e^{\rho a}e^b_\mu\Gamma_{ab}\Phi_{,\rho}\varepsilon = \nabla_\mu\varepsilon + \frac{1}{8}\Gamma^\rho{}_\mu\Phi_{,\rho}\varepsilon.$$

Perform $\Phi \to -\Phi$ transformation in BBS (8.80) supersymmetry transformations for type-I superstring:

$$\delta\Psi_\mu = \nabla_\mu\varepsilon - \frac{1}{8}e^{-\Phi}\tilde{\mathbf{F}}^{(3)}\Gamma_\mu\varepsilon,$$

$$\delta\lambda = -\frac{1}{2}\Gamma^\mu\partial_\mu\Phi\varepsilon + \frac{1}{4}e^{-\Phi}\tilde{\mathbf{F}}^{(3)}\varepsilon,$$

$$\delta\chi = -\frac{1}{2}\mathbf{F}^{(2)}\varepsilon.$$

Then observe that according to definition of the type BBS (8.13) under rescaling of metric the following transformations take place: $\tilde{\mathbf{F}}^{(3)} \to \mathrm{e}^{3\Phi/2}\tilde{\mathbf{F}}^{(3)}$, $\mathbf{F}^{(2)} \to \mathrm{e}^{\Phi}\mathbf{F}^{(2)}$. Therefore SUSY transformations go to

$$\delta\Psi_\mu = \nabla'_\mu\varepsilon - \frac{1}{8}\tilde{\mathbf{F}}^{(3)}\Gamma_\mu\varepsilon,$$

$$e^{-\Phi/2}\delta\lambda = -\frac{1}{2}\Gamma^\mu\partial_\mu\Phi\varepsilon + \frac{1}{4}\tilde{\mathbf{F}}^{(3)}\varepsilon,$$

$$e^{-\Phi}\delta\chi = -\frac{1}{2}\mathbf{F}^{(2)}\varepsilon.$$

This corresponds to BBS (8.85) supersymmetry transformations for heterotic supergravity if one rescales $\lambda \to \lambda e^{\Phi/2}$ and $\chi \to \chi e^{\Phi}$.

`Problem 8.9`

Taub-NUT metric is given by $ds_5^2 = -dt^2 + ds_{TN}^2$, where spatial $4d$ part is

$$ds_{TN}^2 = V(r)\left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) + V^{-1}(r)\left(\frac{R}{2}(d\psi - d\phi) + R\sin^2\frac{\theta}{2}d\phi\right)^2, \qquad (8.77)$$

where $V(r) = 1 + \frac{R}{2r}$, and compactified coordinate y (with period $2\pi R$) is replaced by $\psi = \phi + 2y/R$ with period 4π. Introduce new variable $\rho = \frac{R}{\sqrt{V(r)}}$. As soon as we consider points near $r = 0$, we can calculate metric with accuracy of $\mathcal{O}(r^2)$ (which is equal to $\mathcal{O}(\rho^4)$). Then

$$r^2V(r)\sin^2\theta + V^{-1}(r)\frac{R^2}{4}\cos^2\theta = \frac{rR}{2} + \mathcal{O}(r^2) = \frac{R^2}{4}V^{-1}(r) + \mathcal{O}(r^2) = \frac{\rho^2}{4} + \mathcal{O}(r^2),$$

which is a coefficient of $d\phi^2$ in (8.77). Coefficient of $d\theta^2$ is equal to

$$r^2V(r) = \frac{rR}{2} + \mathcal{O}(r^2) = \frac{\rho^2}{4} + \mathcal{O}(r^2).$$

Finally we observe that $V(r)dr^2 = d\rho^2 + \mathcal{O}(\rho^4)$. As a result from (8.77) in the vicinity of $r = 0$ one goes to

$$ds_{TN}^2 = d\rho^2 + \frac{\rho^2}{4}(d\theta^2 + d\phi^2 + d\psi^2 - 2\cos\theta d\phi d\psi).$$

Determinant of metric corresponding to ds_{TN} is equal to $g = \frac{\rho^6 \sin^2\theta}{64} > 0$. All lead minors are positive too, therefore metric is positive-defined (according to Sylvester criterion), which means it's of Euclidean type. It can be made conformally-flat with obstruction points at $\theta = 0$ (where determinant of (ϕ, ψ)-part of the metric tensor vanishes) and $\rho = 0$.

`Problem` **8.10**

(i) We start with zero-valued antisymmetric 2-form B_2 and $10D$ metric

$$ds^2 = -dt^2 + ds_{TN}^2 + \sum_{i=1}^{5} dx_i^2,$$

where

$$ds_{TN}^2 = V(r)\left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) + V^{-1}(r)\left(dy + R\sin^2\frac{\theta}{2}d\phi\right)^2$$

is $(1+4)D$ Taub-NUT metric. Then we apply the rules of background fields transformation mentioned in the statement of the problem, which are derived in the solution of Problem 6.5. For T-duality in y-direction this gives metric (order $t, r, \theta, \phi, y, \dots$)

$$\tilde{g}_{\mu\nu} = \mathrm{diag}\{-1,\ V(r),\ r^2V(r),\ r^2V(r)\sin^2\theta,\ V(r),\ 1,\ 1,\ 1,\ 1,\ 1\},$$

and antisymmetric 2-form $\tilde{B}_2$ with only non-zero components $\tilde{B}_{y\phi} = -\tilde{B}_{\phi y} = R\sin^2\frac{\theta}{2}$.

(ii) As soon as we've performed a T-duality transformation in compactified direction y, that is orthogonal to 5-brane (which stretches in coordinate directions x^i), then we result in 6-brane in IIB-theory. The metric on this brane is given by $ds_{(6)}^2 = V(r)dy^2 + \sum_{i=1}^5 dx_i^2$.

(iii) Tensions of D-branes support this interpretaion, because unwrapping of one of spatial directions orthogonal to 5-brane will give 6-brane with tension

$$T_{D6} = \frac{T_{D5}}{2\pi\ell_s}. \tag{8.78}$$

Formula BBS (8.114) is an example of calculation of tension for known metric. In that formula integration is performed over spatial coordinated, orthogonal to brane (integration over y coordinate was performed explicitly, giving a factor $2\pi R$). For 6-brane we have one less orthogonal direction than for 5-brane, and if these branes are T-dual, then the direction of their "difference" is compact. Then integral over it gives relation between tensions pointed in (8.78).

`Problem` **8.11**

Consider $D = 11$ M-theory. There're 10 spatial dimensions, therefore number of spatial coordinates which surround $M5$-brane is equal to 4. Then F_4 form may be integrated over a sphere S^4, surrounding $M5$-brane, which is then a source of 'magnetic' charge. This charge is localized on $M5$-brane, therefore on $M5$-brane F_4 form is not exact: $F_4 \neq dA_3$. Then

$$dF_4 = \delta_W,$$

where δ_W is 5-form magnetic current, being a delta-function with support on $M5$-brane. It's defined by the following formula, concerning 6-form field K_6:

$$\int \delta_W \wedge K_6 = \int_{M5} K_6,$$

where integration in the l.h.s. is performed over 11D space-time with 6 coordinates being $M5$-brane world-volume coordinates, integration in the r.h.s. is performed over $M5$-brane world-volume.

6D world-volume theory has Lorentz anomaly, encoded in $6+2=8$ form I_8. Descent equations allow us to reconsider anomaly from the point of local world-volume Lagrangian counterterm, which is proportional to 6-form G_6:

$$I_8 = d\omega_7, \quad \delta\omega_7 = dG_6,$$

where variation δ is assumed under Lorentz transformations. To cancel this anomaly assume interaction of $M5$-brane with some 7-form field ω_7 (which may be expressed through CS forms) in $D=11$, which adds term

$$S_2 = \int F_4 \wedge \omega_7$$

to the action $S_1 = \mu \int F_4 \wedge (\star F)_7 \sim \mu \int d^{11}x |F_4|^2$. Total action is therefore

$$S = S_1 + S_2 = \mu \int F_4 \wedge \star F_7 + \int F_4 \wedge \omega_7,$$

where $\star F_7$ is 7-form Hodge dual to 4-form F_4. Variation of action gives equations of motion:

$$d \star F_7 = -\frac{1}{2\mu} d\omega_7.$$

`Problem` **8.12**

Tension of $D0$-brane is derived in M-theory: it's equal to mass of KK excitation (see BBS (8.106), where according to BBS (8.34) one has $\ell_s = \sqrt{\alpha'}$):

$$T_{D0} = M_{D0} = \frac{1}{R_{11}} = \frac{1}{\ell_s g_s}.$$

Therefore, according to BBS (6.112), (6.114) recursion formulae tension of $D2$-brane is equal to (this result is mentioned in Chapter 6 too, see BBS (6.115))

$$T_{D2} = \frac{1}{(2\pi)^2 \alpha' \ell_s g_s} = \frac{2\pi}{(2\pi \ell_s)^3 g_s}.$$

From the other point, $M2$-brane under dimension reduction of one of its longitudinal coordinates is transformed to fundamental string (because this string is coupled to corresponding B_2-field, obtained by dimensional reduction form A_3), therefore tension of $M2$-brane is given by (see BBS (8.107)):

$$T_{M2} = \frac{T_{F1}}{2\pi R_{11}} = \frac{2\pi}{(2\pi \ell_s)^3 g_s},$$

where we've used the fact, that tension of fundamental string is given by $T_{F1} = \frac{1}{2\pi\alpha'}$ (BBS (2.42)). One easily observes that $T_{D2} = T_{M2}$.

According to BBS (8.22), (8.95) and (8.34) tensions of $M5$ and $NS5$ branes equal to each other. This tension is given by

$$T_{M5} = \frac{2\pi}{(2\pi \ell_p)^6} = \frac{2\pi}{g_s^2 (2\pi \ell_s)^6},$$

therefore tension of wrapped $M5$ brane is equal to

$$2\pi R_{11} T_{M5} = \frac{2\pi}{g_s (2\pi \ell_s)^5},$$

which is equal to the tension of $D4$-brane.

`Problem` **8.13**
BBS (8.131) $\rightarrow$

$$\beta = \sqrt{\frac{2\pi R_{11} T_{M2}}{T_{F1}}}. \tag{8.79}$$

Combine it with BBS (8.136) $\rightarrow$

$$T_{D3} = \frac{T_{M2}}{2\pi R_B \beta^3} = \frac{T_{M2}}{2\pi R_B} \left(\frac{T_{F1}}{2\pi R_{11} T_{M2}} \right)^{\frac{3}{2}}. \tag{8.80}$$

BBS (8.132) $\rightarrow$

$$R_B = \frac{g_s}{(2\pi R_{11})^{3/2} \sqrt{T_{F1} T_{M2}}}.$$

Substitute into (8.80) $\rightarrow$

$$T_{D3} = \frac{T_{F1}^2}{2\pi g_s}. \tag{8.81}$$

Substitute (8.79) and (8.81) into BBS (8.138) $\rightarrow$

$$T_{M5} = \frac{T_{M2}^2}{2\pi} \frac{(2\pi R_{11})^2}{g_s A_M}. \tag{8.82}$$

BBS (8.129) $\rightarrow$ $\mathrm{Im}\tau_M = \mathrm{Im}\tau_B$,
BBS (8.98) $\rightarrow$ $\mathrm{Im}\tau_B = \frac{1}{g_s}$,
therefore BBS (8.126) $\rightarrow$ $A_M = \frac{(2\pi R_{11})^2}{g_s}$.
Substitute into (8.82) $\rightarrow$

$$T_{M5} = \frac{T_{M2}^2}{2\pi}.$$

`Problem` **8.14**
Eq. BBS (8.139) verifies formulae BBS (8.22) for tensions of $M2$- and $M5$-branes. Then

$$T_{M2} T_{M5} = 2\pi \frac{1}{(2\pi)^8 (g_s l_s^3)^3}.$$

BPS bound is defined as $T_{M2} = \alpha \mu_{M2}$, and $T_{M5} = \alpha \mu_{M5}$, where

$$\alpha = \frac{1}{(2\pi)^4 (g_s l_s^3)^{3/2}}.$$

Then Dirac quantization condition (for the minimal product of charges, which is the case here because according to the problem statement both $M2$-brane and $M5$-brane carry one unit of corresponding charges) is indeed satisfied:

$$\mu_{M2} \mu_{M5} = 2\pi.$$

Problem **8.15**

We are going to match M-theory compactified on cylinder M, and $SO(32)$ heterotic string (HO string) compactified on circle. Cylinder has length L_1 in eleventh direction, and circumference L_2 in tenth direction. In HO theory tenth (compactified) direction has length (period) $L_O = 2\pi R_O$.

According to mass formula for compactified closed string, tension of a string is a mass of excitation in compactified direction per unit length (see BBS (6.14) or its generalization BBS (8.123)). Here we consider closed string which wounds compact direction ones, namely consider HO string wrapped on R_O-circle. Its mass is then $2\pi R_O T_1^{(HO)}$. It corresponds to lowest KK mode of $M2$-brane on cylinder. Generally speaking, KK modes on cylinder are described by wave function with periods L_1 and L_2:

$$\psi_{n_1,n_2} \sim \exp\left(2\pi i \left(\frac{n_1}{L_1}x + \frac{n_2}{L_2}y\right)\right),$$

where we have denoted $x = x^{10}$ and $y = x^9$. The mass squared is eigenvalue of $-\partial_x^2 - \partial_y^2$ operator. For KK excitation, corresponding to considered lowest HO string state, mass is equal to $\frac{2\pi}{L_2}$. Matching masses of KK excitations, taking into account factor β, relating metrics in $11D$ and $10D$ ($g^{(M)} = \beta^2 g^{(O)}$, and therefore if we compare some length a, or mass a^{-1}, in HO theory with some length b in M-theory then we are to write it as $a\beta \sim b$), one gets

$$\beta L_2 R_O T_1^{(HO)} = 1. \tag{8.83}$$

Now lets match compactified HO string, with lowest mass input from compactified direction being $\frac{1}{R_O}$, and compactified M-theory with lowest mass input from compactified directions being $A_M T_{M2} = L_1 L_2 T_{M2}$ (again, use appropriately β factor):

$$L_1 L_2 T_{M2} = \frac{2\pi\beta}{L_O}. \tag{8.84}$$

Equations (8.83) and (8.84) together give us

$$L_1 L_2^2 T_{M2} = \frac{(2\pi)^2}{T_1^{(HO)} L_O^2}.$$

Also from (8.83) and (8.84) one can also easily figure out, that

$$\beta = \sqrt{\frac{L_1 T_{M2}}{T_1^{(HO)}}}, \tag{8.85}$$

and

$$L_0 = \frac{2\pi\beta}{L_1 L_2 T_{M2}}. \tag{8.86}$$

Now observe that $(1,0)$ $D5$-brane in heterotic string theory (actually heterotic string is closed string theory, therefore Dirichlet brane is not appropriate term) is 'replaced' by $T_5^{(HO)}$-brane, which is T_{M5}-brane without x^{11} as longitudinal direction. If we compactify $T_5^{(HO)}$-brane on circle of radius R_O and T_{M5}-brane on L_2-line interval, we will get (with respect to β^5 factor, denoting matching of 5 dimensions of length) matching of tensions of 4-branes:

$$L_O\beta^5 T_5^{(HO)} = L_2 T_{M5}.$$

Using (8.85), (8.86) and BBS (8.139) for the relation $\frac{T_{M5}}{(T_{M2})^2}$, we get

$$T_5^{(HO)} = \frac{(T_1^{(HO)})^3}{(2\pi)^2}\left(\frac{L_2}{L_1}\right)^2.$$

Finally let's match masses of KK excitations of M-theory and HO-string, as it was done in the beginning of this solution, but using formula $|n_2 - n_1\tau_B|T_1^{(HO)}\beta$ for tension (mass per unit length in compactified direction) for mass of (n_1, n_2) KK excitation of HO string (β factor makes our consideration 11D). Namely we are interested in $(1, 0)$ KK excitation. Because $\tau_B = \frac{i}{g_s}$ (in difference with BBS (8.98), presented for type-IIB superstring, HO string as type-I string has no C_0 fileds - there's 'just' C_2), according to presented above mass of M-theory KK excitation (applied for $(n_1, n_2) = (1, 0)$ case), we obtain

$$L_O\frac{1}{g_s}T_1^{(HO)}\beta = \frac{2\pi}{L_1}.$$

Combining this with (8.83) one gets

$$g_s = \frac{L_1}{L_2}.$$

9 String geometry

`Problem` **9.1**

According to BBS (9.241) dual form on four-dimensional manifold is determined by formula

$$\star(dx^\mu \wedge dx^\nu) = \frac{1}{2}E^{\mu\nu\lambda\rho}g_{\lambda\lambda'}g_{\rho\rho'}dx^{\lambda'} \wedge dx^{\rho'},$$

where $E^{\mu\nu\lambda\rho} = \frac{\varepsilon^{\mu\nu\lambda\rho}}{\sqrt{|g|}}$.

We deal with rectangular torus T^4 with metric having only non-zero components $g_{z_1\bar{z}_1} = g_{\bar{z}_1 z_1} = g_{z_2\bar{z}_2} = g_{\bar{z}_2 z_2} = \frac{1}{2}$, and therefore $\sqrt{|g|} = \frac{1}{4}$. We also hold the definition $\varepsilon^{12\bar{1}\bar{2}} = 1$. As a result one has

$$\star(dz^1 \wedge dz^2) = 4g_{\bar{1}1}g_{\bar{2}2}dz^1 \wedge dz^2 = dz^1 \wedge dz^2,$$
$$\star(d\bar{z}^{\bar{1}} \wedge d\bar{z}^{\bar{2}}) = d\bar{z}^{\bar{1}} \wedge d\bar{z}^{\bar{2}},$$
$$\star(dz^1 \wedge d\bar{z}^{\bar{2}}) = -dz^1 \wedge d\bar{z}^{\bar{2}},$$
$$\star(dz^2 \wedge d\bar{z}^{\bar{1}}) = -dz^2 \wedge d\bar{z}^{\bar{1}},$$
$$\star(dz^1 \wedge d\bar{z}^{\bar{1}} + dz^2 \wedge d\bar{z}^{\bar{2}}) = dz^1 \wedge d\bar{z}^{\bar{1}} + dz^2 \wedge d\bar{z}^{\bar{2}},$$
$$\star(dz^1 \wedge d\bar{z}^{\bar{1}} - dz^2 \wedge d\bar{z}^{\bar{2}}) = -(dz^1 \wedge d\bar{z}^{\bar{1}} - dz^2 \wedge d\bar{z}^{\bar{2}}).$$

In addition note that 3 anti self-dual 2-forms provided here should be accompanied with 16 anti self-dual 2-forms J of the type BBS (9.25) from 16 EH spaces around 16 singularities of orbifold. Observe also that among formulae above one has 3 self-dual 2-forms.

Finally note that considered forms are harmonic, or equivalently, they form non-trivial basic elements of de Rahm cohomology class H^2. Indeed, if the form is closed and self-dual (or anti self-dual), then it's obviously harmonic. Forms J are closed as Kähler forms, and they are anti self-dual, as it's shown in the solution of Problem 9.2. Other six forms considered above are both closed and (anti) self-dual. Therefore they are all harmonic.

`Problem` **9.2**

The positive-oriented order of coordinates is (r, ψ, θ, ϕ), as it's shown later it gives positive determinant of metric tensor. Then from BBS (9.24) it follows, that metric tensor of EH space is given by matrix:

$$g_{\mu\nu} = \begin{bmatrix} \frac{1}{\Delta} & 0 & 0 & 0 \\ 0 & \frac{r^2\Delta}{4} & 0 & \frac{r^2\Delta\cos\theta}{4} \\ 0 & 0 & \frac{r^2}{4} & 0 \\ 0 & \frac{r^2\Delta\cos\theta}{4} & 0 & \frac{r^2}{4}(\Delta\cos^2\theta+\sin^2\theta) \end{bmatrix}_{\mu\nu}$$

Then determinant of metric tensor is equal to $g = \frac{r^6\sin^2\theta}{64}$. Spherical coordinate θ takes values in the range $[0, \pi]$, hence $\sqrt{g} = \frac{r^3\sin\theta}{8} > 0$. The quantity $E^{\mu\nu\lambda\rho} = \frac{\varepsilon^{\mu\nu\lambda\rho}}{\sqrt{g}}$ is a tensor. According to BBS (9.241) the Hodge dual forms are calculated with the help of formula

$$\star(dx^\mu \wedge dx^\nu) = \frac{1}{2}E^{\mu\nu\lambda\rho}g_{\lambda\lambda'}g_{\rho\rho'}dx^{\lambda'} \wedge dx^{\rho'}.$$

The factor $\frac{1}{2}$ will be compensated after summation of (λ, ρ) indices. Due to positive-oriented order of coordinates mentioned above (and due to the fact that in Euclidean space we can rise all four indices of ε-symbol without change of the sign; while orientation is defined namely by antisymmetric tensor with lower indices - volume form components) we have $\varepsilon^{r\psi\theta\phi} = 1$. Then we get:

$$\star(dr \wedge d\psi) = -\frac{r}{2\sin\theta}\left((\sin^2\theta + \Delta\cos^2\theta)d\phi + \Delta\cos\theta d\psi\right) \wedge d\theta;$$

$$\star(dr \wedge d\phi) = -\frac{r\Delta}{2\sin\theta}d\theta \wedge (d\psi + \cos\theta d\phi);$$

$$\star(d\theta \wedge d\phi) = \frac{2}{r\sin\theta}dr \wedge (d\psi + \cos\theta d\phi).$$

Hence from BBS (9.25) it follows that

$$\star J = \frac{r^2}{4}\sin\theta d\theta \wedge d\phi - \frac{r}{2}dr \wedge d\psi - \frac{r}{2}\cos\theta dr \wedge d\phi.$$

Observe, that $\star J = -J$, that is Kähler form of EH space is anti self-dual.

`Problem` **9.3**

2-sphere S^2 is a Riemann surface, which is complex projective space CP^1. The later is Kähler space with Kähler potential, explored for general CP^n in the solution of the Problem 9.4. For the first map of CP^1, with all points except infinitely distant point on complex plane one has Kähler potential

$$K = \ln(1 + z\bar{z}).$$

So, the metric tensor on sphere has only two non-zero components, which due to symmetry of metric on hermitian manifold (which Kähler manifold is) are equal to each other:

$$g_{z\bar{z}} = g_{\bar{z}z} = \partial_z \partial_{\bar{z}} K = \frac{1}{(1+z\bar{z})^2}.$$

A known fact is that on a Kähler manifold Ricci tensor is given by components

$$R_{\bar{b}c} = \partial_{\bar{b}} \partial_c \ln \sqrt{g},$$

which is here reduced to only one non-zero component

$$R_{\bar{z}z} = R_{z\bar{z}} = -\frac{2}{(1+z\bar{z})^2}.$$

In analogous manner to Kähler form $k_{a\bar{b}} = J_a^c g_{c\bar{b}}$, defined through Kähler metric and complex structure tensor (together with $k_{\bar{a}b} = -J_{\bar{a}}^{\bar{c}} g_{b\bar{c}}$), we can naturally define Ricci form with components antisymmetric in indices permutation. As a result we get Ricci form (see BBS (9.278))

$$\mathcal{R} = -2i \frac{dz \wedge d\bar{z}}{(1+z\bar{z})^2}.$$

Let's proceed to Chern class. First we integrate Ricci form over complex plane, going to real coordinates and then to plane polar coordinates:

$$\int \mathcal{R} = -4 \int \frac{dx \wedge dy}{(1+x^2+y^2)^2} = -8\pi \int \frac{rdr}{(1+r^2)^2} = -4\pi.$$

Then according to BBS (9.279) one has $c_1 = -2$.

Problem 9.4

Let's go a little bit different way from the one proposed on p. 369 BBS. Introduce Kähler potential

$$K(z^a, \bar{z}^{\bar{a}}) = \ln \sum_{a=1}^{n+1} z^a \bar{z}^{\bar{a}}$$

before choosing an open set with some concrete non-zero z^a. Observe that under CP^n identification $(z^1, \cdots, z^{n+1}) \sim (\lambda z^1, \cdots, \lambda z^{n+1})$ this potential changes by constant shift by $\ln \lambda$, which will not affect metric. As a consequence note that infinitesimally close to the point $\{z^a\}$ point $\{\lambda z^a\}$, where $\lambda = 1 + \varepsilon,\ \varepsilon \to 0$ have Kähler potential, which differ by coordinate-independent constant (it holds for any λ, not necessary close to 1, but we want to find ds for equivalent by λ-identification close points), and therefore the second differential between these values is essentially zero, that signifies the zero value of distance between equivalent points:

$$\Delta_b \Delta_{\bar{c}} K = \Delta_b \left[\ln \left(\sum_{a=1}^{n+1} z^a \bar{z}^{\bar{a}} |1+\varepsilon|^2 \right) - \ln \sum_{a=1}^{n+1} z^a \bar{z}^{\bar{a}} \right] = \Delta_b \ln |1+\varepsilon|^2 = 0.$$

Then the metric on CP^n is given by

$$ds^2 = \partial \bar{\partial} K =$$

$$= (\bar{z}^{\bar{f}} z^f)^{-1} \left(dz^a - z^a \frac{\bar{z}^{\bar{b}} dz^b}{z^c \bar{z}^{\bar{c}}} \right) \left(d\bar{z}^{\bar{a}} - \bar{z}^{\bar{a}} \frac{z^d d\bar{z}^{\bar{d}}}{z^c \bar{z}^{\bar{c}}} \right),$$

where we've made some additional step to get to this view of interval. Then on Kähler manifold on each open submanifold with non-zero z^a (with some particular a) one uses CP^n identification condition to set $z^a = 1$, which in text (BBS p. 369) is described as transition to variables $w^b = \frac{z^b}{z^a}$, $b \neq a$. It's naturally the same, up to a shift of Kähler potential by $\ln z^a + \ln \bar{z}^{\bar{a}}$ (this shift - Kähler transformation - will not change the metric). Anyway, the Kähler potential will be

$$K = \ln \left(1 + \sum_{b \neq a} z^b \bar{z}^{\bar{b}} \right).$$

Transition to other open sets (with different non-zero coordinate) will just shift Kähler potential by a constant according to pointed above. One can also perform a permutation of coordinates, to keep fixed coordinate always called z^{n+1}.

`Problem` **9.5**

First of all note that if $\tau = \tau_1 + i\tau_2$ is a modular parameter of torus T^2, then one can choose two real coordinates σ_1, σ_2 (also denoted as x and y in the solution of Ex. 9.9, which explores *complex structure* moduli space) with periods equal to 1, and define metric tensor, with determinant equal to 1:

$$g = \frac{1}{\tau_2} \begin{pmatrix} \tau_1^2 + \tau_2^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix}.$$

This procedure was already mentioned in the solution of Problem 7.9. If one considers complex coordinates z, $\bar{z}$ with $z = \sigma_2 + \tau\sigma_1$ (and corresponding identifications $z \sim z + 1$, $z \sim z + \tau$), one will get the length element

$$ds^2 = 2g_{z\bar{z}} dz d\bar{z}$$

with $g_{z\bar{z}} = \frac{1}{2\tau_2}$ (correspondingly $g^{z\bar{z}} = 2\tau_2$). Kähler form components are then: $J_{z\bar{z}} = i\frac{1}{2\tau_2}$ and $J_{\bar{z}z} = -i\frac{1}{2\tau_2}$, Kähler form is given by equation

$$J = J_{z\bar{z}} dz \wedge d\bar{z} = \frac{i}{2\tau_2} dz \wedge d\bar{z}.$$

Introduce also antisymmetric B moduli field, being $(1,1)$-form:

$$B = ib dz \wedge d\bar{z}.$$

This field has only one independent component, and was already employed in the solution of Problem 7.12. Here it plays its role in complexification of Kähler form: $\mathcal{J} = B + iJ$. General expression BBS (9.127) for expansion of complexified Kähler form through basis $(1,1)$-forms now reduces to just one basis $(1,1)$-form being $dz \wedge d\bar{z}$ and correspondingly just one complex coordinate w:

$$\mathcal{J} = w dz \wedge d\bar{z}.$$

Decompose $w = w_1 + iw_2$. Then

$$\mathcal{J} - \bar{\mathcal{J}} = 2iJ = -\frac{1}{\tau_2} dz \wedge d\bar{z} =$$

$$= 2w_1 dz \wedge d\bar{z},$$

therefore one concludes that $w_1 = -\frac{1}{2\tau_2}$,

$$J = -iw_1 dz \wedge d\bar{z}.$$

and

$$\delta J = -i(dw_1)dz \wedge d\bar{z},$$

from which it follows that

$$\delta g_{z\bar{z}} = -dw_1.$$

Consider next

$$\mathcal{J} + \bar{\mathcal{J}} = 2B = 2ibdz \wedge d\bar{z} =$$

$$= 2iw_2 dz \wedge d\bar{z},$$

therefore one concludes that $w_2 = b$,

$$B = iw_2 dz \wedge d\bar{z}.$$

and

$$\delta B = i(dw_2)dz \wedge d\bar{z},$$

from which it follows that

$$\delta B_{z\bar{z}} = idw_2.$$

Using second part of formula BBS (9.97) for Kähler structure moduli space metric we obtain

$$ds^2 = \frac{1}{2V} \int g^{a\bar{b}} g^{c\bar{d}} (\delta g_{a\bar{d}} \delta g_{c\bar{b}} - \delta B_{a\bar{d}} \delta B_{c\bar{b}}) \sqrt{g} d^2 z =$$

$$= 2\tau_2^2 (dw_1^2 + dw_2^2) = \frac{1}{2w_1^2} (dw_1^2 + dw_2^2),$$

where $V = \int d^2 z \sqrt{g}$.

Let's propose that Kähler structure moduli space potential is given by

$$\mathcal{K} = -\frac{1}{4} \ln \left(\int J \right) = -\frac{1}{4} \ln \left(\frac{i}{2\tau_2} \int dz \wedge d\bar{z} \right).$$

Due to pointed above it follows that Kähler metric tensor has the sole non-zero component which equals to

$$\frac{\partial^2 \mathcal{K}}{\partial w \partial \bar{w}} = \tau_2^2 = \frac{1}{4w_1^2}.$$

Indeed, for example it was used that

$$-4\frac{\partial \mathcal{K}}{\partial \bar{w}} = \frac{-\frac{i}{2\tau_2^2} \int dz \wedge d\bar{z}}{\frac{i}{2\tau_2} \int dz \wedge d\bar{z}} \frac{\partial \tau_2}{\partial \bar{w}} = -\frac{1}{\tau_2} \left(\frac{\partial w_1}{\partial \tau_2} \right)^{-1} = -2\tau_2.$$

Corresponding interval on Kähler structure moduli space is equal to

$$ds^2 = 2\partial\bar{\partial}\mathcal{K} dw d\bar{w} = \frac{1}{2w_1^2} dw d\bar{w} = \frac{1}{2w_1^2} (dw_1^2 + dw_2^2).$$

Problem **9.6**

The solution of this problem is heavily based on the solution of Problem 7.12, mainly its third and fourth parts (fourth part is actually the solution of the present problem itself, while third part explains some details and introduces some definitions). Let's make the connection with pointed on p. 388 BBS. Formula BBS (9.87) is written for rectangular torus with no B-field. Indeed, according to Problem 7.12 we can use reparametrization invariance of torus surface to eliminate two fields on it. We choose $G_{12} \sim \theta$ component of metric to eliminate, together with B_{12} - the sole non-zero component of B-field. Then complex parameters τ and ρ (introduced in the solution of Pr. 7.12), describing torus, reduce to

$$\tau = i\frac{\sqrt{\det G}}{G_{22}}, \quad \rho = i\sqrt{\det G}.$$

Here $G = \mathrm{diag}\{R_1^2,\ R_2^2\}$ is diagonal metric of rectangular torus with periods R_1 and R_2, and therefore these formulae are of the form BBS (9.87), with accuracy of redefinition $R_1 \leftrightarrow R_2$, as it's easy to check. Then mirror symmetry $\tau \leftrightarrow \rho$ as explained in the solution of Pr. 7.12 is: 1) always reduced to that on rectangular torus; 2) is a duality transformation, which therefore doesn't change spectrum of the string.

Problem **9.7**

We are going to start the solution of this problem with procedure of fixation of metric gauge. The thing is that metric transformations which can be represented as $\delta g_{mn} = \nabla_m \xi_n + \nabla_n \xi_m$ are actually transformations of metric associated with change of coordinates $\delta x_m = \xi_m$. Therefore, $D = 6$ independent transformations of metric are purely gauge. We can fix them by imposing m constraints

$$\Gamma^m = g^{kn}\Gamma^m_{kn} = 0, \tag{9.87}$$

which always may be satisfied by proceeding to appropriate coordinate system (by definition metric g^{kn} on the r.h.s. of (9.87) is initial Ricci-flat Kähler metric, while connections Γ are constructed with the varied metric $g + \delta g$). As soon as coordinate system is fixed, no metric changes produced by more coordinate changes will be allowed. Indeed, (9.87) may be rewritten in the form (of BBS (9.91)):

$$\nabla^m \delta g_{mn} - \frac{1}{2}\nabla_n \delta g^k_k = 0, \tag{9.88}$$

where g_{mn} is our initial Ricci-flat Kähler metric, which is used to construct connections and covariant derivatives (the simplest way to prove this is to imagine, that our initial Ricci-flat metric is actually flat, at least locally: $g_{mn} = \eta_{mn} = \delta_{mn}$, which will highly simplify all formulae with derivative of metric, expanded to the first order of δg, for example, affine connections would be $\Gamma^m_{nk} = \frac{1}{2}\eta^{mp}(h_{pn,k} + h_{pk,n} - h_{nk,p})$, and then return to real metric g_{mn}, changing all partial derivatives to covariant derivatives and assuming that indices are getting up and down with the help of our initial Ricci-flat Kähler metric g^{mn}). If one conveniently defines $h_{mn} = \delta g_{mn}$, then one will get rational explanation of definition of δg^k_k, provided after BBS (9.91): $\delta g^k_k = g^{km}h_{mk}$. If one then wants to satisfy (9.88), one first of all performs coordinate change to transform $D = 6$ independent components of metric: $h_{mn} = \nabla_m \xi_n + \nabla_n \xi_m$:

$$\nabla^2 \xi_m = \nabla_n h^n_m - \frac{1}{2}\nabla_m h^n_n.$$

Having done this, one is able to perform actual (non-presentable as result of coordinate change) moduli transformations of metric. Doing this in a locally flat coordinate system, and performing transition to arbitrary coordinate system, as described above, one gets formula BBS (9.92).

`Problem` **9.8**

Consider $(2,1)$-form, given by BBS (9.96):

$$\eta = \Omega_{abc} g^{c\bar{d}} \delta g_{\bar{d}\bar{e}} dz^a \wedge dz^b \wedge d\bar{z}^{\bar{e}}.$$

This form will be proved to be hamonic if one proves that $\delta g_{\bar{d}\bar{e}}$ are elements of harmonic form, because 3-form Ω is harmonic form of CY_3, and Kähler form $J = i g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$ is closed.

Suppose now we perform a variation of complex structure of CY manifold. CY is complex space, and therefore it possesses integrable complex structure: we can diagonalize complex structure tensor on the whole manifold. In terms of complex coordinates it will be written as $J^a_b = i\delta^a_b$ (in holomorphic coordinates) and $J^{\bar{a}}_{\bar{b}} = -i\delta^{\bar{a}}_{\bar{b}}$ (in antiholomorphic coordinates). We are interested in variations of complex structure $\tilde{J}^m_n = J^m_n + \tau^m_n$ (we are using real indices, because for complex indices we are to determine whether they are holomorphic or not). We assume that variation of complex structure still leaves manifold complex. Then two conditions must be satisfied: first - condition of almost complex structure $\tilde{J}^a_b \tilde{J}^b_c = -\delta^a_c$ (and the same for antiholomorphic case). So, for both (anti)holomorphic indices of variation of complex structure tensor one has condition $\tau^a_b = 0$, $\tau^{\bar{a}}_{\bar{b}} = 0$ (for infinitesimal τ). Therefore, the only possible case for moduli variations of complex structure is constructed with the help of τ with indices of different types. Second condition is that modified complex structure still must be integrable, which is equivalent to vanishing of Nijenhuis tensor. Because of we started with complex manifold, and therefore N-tensor was zero, then we must require vanishing of variation of N-tensor. To compute this variation note that terms of the type $\partial_{[q} J^p_{n]}$ in the expression for N-tensor vanish when they are not varied, because they will be proportional to derivative of delta-symbol. Then, index structure of variation of N-tensor is different from index structure of initial N-tensor, because variation of complex structure has index structure different from that of starting complex structure. To determine this index structure more concretely one should take into account the fact that only non-zero terms of variation of N-tensor are of the type $J\partial\delta J \sim J\partial\tau$, where J has indices of the same type and τ has indices of opposite types. Then one gets the following non-zero variations

$$\delta N^a_{\bar{b}\bar{c}} = -i(\partial_{\bar{b}} \tau^a_{\bar{c}} - \partial_{\bar{c}} \tau^a_{\bar{b}}),$$

and complex conjugate to this. Because complex structure $\tilde{J}$ is a tensor, then $\tau^a_{\bar{b}}$ is a tensor too, then τ^a may be viewed as antiholomorphic form (as $(0,1)$-form). Vanishing of variation of N-tensor then means $\bar{\partial}\tau^a = 0$, that is $\tau^a \in H^{0,1}$.

Because of $g_{mn} = J^l_m k_{ln}$, when J is infinitesimally varied, Kähler form k shouldn't be varied, and therefore variation of metric will have index structure, opposite to that of initial metric. Therefore terms of the type δg_{ab} and $\delta g_{\bar{a}\bar{b}}$ have their origin in variation of complex structure and are harmonic, because variation of complex structure is harmonic.

`Problem` **9.9**

If one represents complex manifold M with 3 complex dimensions in terms of unification of sets $(A^I,\ B_I)$, where A^I and B_I are $(2,1)$ and $(1,2)$ homology basic cycles, then to integrate antisymmetrized product $\alpha\wedge\beta$ of $(2,1)$ and $(1,2)$ forms form over manifold one may use formula BBS (9.116).

According to BBS (9.104), one has

$$e^{-\mathcal{K}^{2,1}} = i\int \Omega\wedge\bar{\Omega}.$$

With the help of BBS (9.107), (9.109), (9.116) one easily proves BBS (9.117).

`Problem` **9.10**

Formula BBS (9.132) for Kähler potential on moduli space of CY Kähler structure $\mathcal{J}$ will agree with the formula BBS (9.129) for the same matter, if they differ by constant multiplier. Indeed, in that case Kähler potential $\mathcal{K}^{1,1}$ will be just shifted by a constant, which will have no affection on metric. Observe, that for real basis $\{e_\alpha\}$ of $(1,1)$-forms in the space of moduli transformations of Kähler structure one has formula BBS (9.127), which gives expressions $e_\alpha = \frac{\partial\mathcal{J}}{\partial w^\alpha}$, and $e_\alpha = \frac{\partial\bar{\mathcal{J}}}{\partial\bar{w}^\alpha}$ (remember that $(1,1)$-forms are real). Use these expressions while differentiating prepotential $G(w)$, given by BBS (9.131). For example:

$$\sum_{\alpha=1}^{h^{1,1}} \bar{w}^\alpha \frac{\partial G(w)}{\partial w^\alpha} = \frac{1}{2w^0}\int \bar{\mathcal{J}}\wedge\mathcal{J}\wedge\mathcal{J}.$$

Complex conjugation gives second set of summarized terms in r.h.s. of BBS (9.132). This will give

$$i\sum_{\alpha=1}^{h^{1,1}}\left(w^\alpha\frac{\partial\bar{G}(\bar{w})}{\partial\bar{w}^\alpha} - \bar{w}^\alpha\frac{\partial G(w)}{\partial w^\alpha}\right) =$$
$$= \int J\wedge J\wedge J + \int B\wedge J\wedge B.$$

At the same time one has auxiliary term w^0 in the sum:

$$i\left(w^0\frac{\partial\bar{G}(\bar{w})}{\partial\bar{w}^0} - \bar{w}^0\frac{\partial G(w)}{\partial w^0}\right) =$$
$$= -\int B\wedge J\wedge B.$$

Altogether one has

$$i\sum_{A=0}^{h^{1,1}}\left(w^A\frac{\partial\bar{G}(\bar{w})}{\partial\bar{w}^A} - \bar{w}^A\frac{\partial G(w)}{\partial w^A}\right) = \int J\wedge J\wedge J,$$

which according to pointed above agrees with BBS (9.129).

In this solution it was used the fact, that for for 2-forms it takes place $B\wedge J = J\wedge B$.

`Problem` **9.11**

From BBS (9.117) formula for Kähler potential of complex-structure moduli space of CY manifold and formula BBS (9.142) for 'dual' singular first coordinate on this moduli space, it follows formula BBS (9.143), describing corresponding term in Kähler potential:

$$\mathcal{K} \sim \ln\left(|X^1|^2 \ln|X^1|^2\right),$$

which allows us to find corresponding component $G_{1\bar{1}}$ of metric of moduli space:

$$G_{1\bar{1}} = \partial_1 \partial_{\bar{1}} \mathcal{K} \sim \frac{1}{|X^1|^2 (\ln|X^1|^2)^2},$$

which is singular if $X^1 \to 0$, because power function is faster, then logarithm, as, b.t.w., may be easily demonstrated using the fact, that limit of fraction of smooth functions is equal to limit of fraction of their derivatives.

`Problem` **9.12**

Technics applied here are similar to that used for exploration of κ-symmetry of GS superstring in Chapter 5. We are going to start with derivation of useful identity on Γ-matrices, with the help of simple intuitive method like that used in the solution of Problem 5.6. Such a method is independent of dimension of space-time, then it doesn't matter that we were dealing with ten-dimensional superstring theory in Ch. 5 and we are dealing with eleven-dimensional M-theory now. The formula to be derived is

$$\begin{aligned}
&\{\Gamma_{MNP}, \Gamma_{M'N'P'}\} = \\
&= -2(\eta_{MM'}\eta_{NN'}\eta_{PP'} - \\
&-\eta_{MM'}\eta_{NP'}\eta_{PN'} + \\
&+\eta_{MP'}\eta_{NM'}\eta_{PN'} - \\
&-\eta_{MP'}\eta_{NN'}\eta_{PM'} + \\
&+\eta_{MN'}\eta_{NP'}\eta_{PM'} - \\
&-\eta_{MN'}\eta_{NM'}\eta_{PP'}) + \\
&+2\Gamma_{MNPM'N'P'}.
\end{aligned} \tag{9.89}$$

This formula may be proved by substitution of particular sets of $((M,N,P),\ (M',N',P'))$ to be $((0,1,2),\ (0,1,2))$, $((0,1,2),\ (0,2,1))$, and so on, to make one of first six terms in r.h.s. of (9.89) non-zero and others zero (the last term will be zero too due to antisymmetrization of equal indices).

Now, define operator γ through BBS (9.146) and

$$P_\pm = \frac{1}{2}\left(1 \pm \frac{i}{6}\gamma\right).$$

To explore features of projector operators, one shall find γ^2:

$$\gamma^2 = (\varepsilon^{\alpha\beta\gamma}\partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P)(\varepsilon^{\alpha'\beta'\gamma'}\partial_{\alpha'} X^{M'} \partial_{\beta'} X^{N'} \partial_{\gamma'} X^{P'})\Gamma_{MNP}\Gamma_{M'N'P'}, \tag{9.90}$$

namely one must show, that $\gamma^2 = -36$. It's convenient to represent $\gamma^2 = \frac{1}{2}\{\gamma,\gamma\}$, to get anticommutator of Γ_{MNP} matrices and use formula (9.89). Then note that the last term in r.h.s. of (9.89) doesn't contribute to γ^2, because $\Gamma_{MNPM'N'P'}$ is antisymmetric in permutations of (M,N,P) and (M',N',P'), but two multipliers in round brackets in (9.90) are symmetric. Finally note that according to definition of determinant, for matrix $G_{\alpha\beta}=\eta_{MN}\partial_\alpha X^M\partial_\beta X^N$ one has determinant G satisfying equation

$$3!G=\varepsilon^{\alpha\beta\gamma}\varepsilon^{\alpha'\beta'\gamma'}G_{\alpha\alpha'}G_{\beta\beta'}G_{\gamma\gamma'}.$$

The square root of this determinant is assumed to be present in denominator of tensor $\varepsilon^{\alpha\beta\gamma}$. Therefore, when we calculate γ^2, due to (9.89) we get six equal to each other terms $-2\cdot 3!$. Factor 2 compensates factor $\frac{1}{2}$ from $\frac{1}{2}\{\gamma,\gamma\}$. Therefore what remains is $\gamma^2=-36$.

The rest of the solution is obvious, because now it's evidently that $(P_+)^2=P_+$, $(P_-)^2=P_-$, $P_+P_-=0$.

Problem 9.14

Moduli variations of complex structure should lead to tensor which still will be complex structure and which will lead to vanishing of Nijenhuis tensor. Therefore it may be shown that variations of complex structure are given by tensor $\tau_a^{\bar{b}}$, which is a closed $(1,0)$-form for each value of index $\bar{b}$. The proof was presented in the soluion of Problem 9.8.

With the help of metric tensor we can make both indices of τ lower and denote the resulting tensor as Ω_{ab} to make contact with BBS (9.172). Because of metric tensor on Kähler manifold is expressed through Kähler form k and complex structure J in a way $g_{mn}\sim k_{mk}J^k_n$, then variation of metric, which follows from variation of complex structure, is given by

$$\delta g_{ab}=\Omega_{ac}g^{c\bar{d}}k_{b\bar{d}}+(a\leftrightarrow b),$$

where $g^{c\bar{d}}$ is introduced to make covariant contractions of indices, and $a\leftrightarrow b$ is used to make variation of metric symmetric in permutation of indices. Because of on Kähler manifold, which is a complex manifold, one has $J^a_b=i\delta^a_b$, then $k_{a\bar{b}}=ig_{a\bar{b}}$, and therefore

$$\delta g_{ab}\sim i\Omega_{ac}g^{c\bar{d}}g_{b\bar{d}}+i\Omega_{bc}g^{c\bar{d}}g_{a\bar{d}}=i\Omega_{ab}+i\Omega_{ba}=0.$$

Observe the difference with three-dimensional CY manifolds, where there're no $(2,0)$-forms.

Problem 9.15

In $D=6$ space-time anomaly form is I_8, as explained on page 173 BBS. To cancel anomalies we should be able to introduce local counterterm of the type BBS (5.135), where in our particular case Y_8 8-form is to be replaced by Y_4 4-form. Returning to our $(D+2)$-anomaly form we conclude, using descent equations BBS (5.106), (5.107), that anomaly form I_8 has to be factorizable into two 4-form parts.

From equations BBS (5.112), (5.114), (5.116), and equations provided in the solution of Ex. 5.9 for fermionic anomaly forms, one summarizes that *non-factorizable* terms of 8-forms are

$$I_A(R)=\frac{7}{8\cdot 180}\mathrm{tr}R^4 \qquad \text{self-dual tensor;} \tag{9.91}$$

$$I_{1/2}(R)=\frac{1}{32\cdot 180}\mathrm{tr}R^4 \qquad \text{left-handed Weyl spinor;} \tag{9.92}$$

$$I_{3/2}(R) = \frac{49}{128 \cdot 9} \mathrm{tr} R^4 \qquad \text{left-handed gravitino.} \tag{9.93}$$

Chiral content of $\mathcal{N} = 2$, $D = 6$ type-II supergravity is five self-dual antisymmetric tensors and two left-handed gravitino. Chiral content of tensor multiplet of the same supersymmetry is anti-self-dual antisymmetric tensor and two right-handed Weyl spinors. Therefore for 21 tensor multiplets and supergravity multiplet non-factorizable 8-form term is

$$-16 I_A(R) + 2 I_{3/2}(R) - 42 I_{1/2}(R). \tag{9.94}$$

With the help of (9.91), (9.92) and (9.93) one easily shows, that 8-form (9.94) cancels out.

`Problem` **9.16**

Kinetic term of τ field of the action BBS (9.189) on complex plane in terms of conformal coordinates is

$$S_\tau = 4 \int \frac{\partial \bar{\tau} \bar{\partial} \tau + \partial \tau \bar{\partial} \bar{\tau}}{(\tau - \bar{\tau})^2} d^2 z.$$

Here conformal factor is eliminated from the action, because its inverse from $g^{z\bar{z}}$ cancels out with that from $\sqrt{|g|}$. Lagrange equations obviously give:

$$\partial \bar{\partial} \bar{\tau} + \frac{2 \partial \bar{\tau} \bar{\partial} \bar{\tau}}{\tau - \bar{\tau}} = 0.$$

Obviously for holomorphic τ this variation vanishes.

`Problem` **9.18**

Consider 3-form

$$\phi = dy^{123} + dy^{145} + dy^{167} + dy^{246} - dy^{257} - dy^{347} - dy^{356},$$

where fundamental 3-forms are

$$dy^{ijk} = dy^i \wedge dy^j \wedge dy^k.$$

The form ϕ is obviously preserved by action of α, β and γ transformations, given by equations BBS (9.215)-(9.217). It's also apparently that $\alpha^2 = \beta^2 = \gamma^2 = 1$ and that different transformations commute with each other.

Let's study fixed points of α, acting on T^7. First of all notice that coordinates $(y^1,\ y^2,\ y^3)$ are all fixed (similar triplets exist for β and γ too) and form subspace T^3. The rest subspace of T^7 is $T^4 = T^2 \times T^2$, where each T^2 may be covered by complex coordinate: one with $z^1 = y^4 + iy^5$ and the other one with $z^2 = y^6 + iy^7$. Then observe that α has on these tori fixed points when one of complex coordinates z_a equals to one of four values 0, $\frac{1}{2}$, $\frac{i}{2}$, $\frac{1+i}{2}$ (these are all fixed points on torus, which are not equivalent to each other by torus similarity factorization of complex plane). Therefore there're $4^2 = 16$ fixed points of $T^2 \times T^2$ and therefore fixed space of α action on T^7 is 16 copies of T^3. Exactly the same fixed points of $T^4 = \frac{T^7}{T^3}$ with fixed T^3 as for α take place for β and γ, therefore each of them has fixed tori T^3 too. At the same time, points which are fixed for α are not fixed for β and γ, and so on. The crucial fact here is presence of $y \to \frac{1}{2} - y$ transformations.

Problem **9.19**

Consider 3-cycle with coordinates σ^α, $\alpha = 1,2,3$ on it in $D = 7$ manifold with G_2 holonomy. Then according to BBS (9.144)-(9.148), condition of supersymmetry preservation by this 3-cycle is given by equation BBS (9.218). Here we study 3-cycle in M_7, not $M_4 = M_{11}/M_7$, therefore with proper indices BBS (9.218) is to be rewritten as

$$P_-\eta = \frac{1}{2}\left(1 - \frac{i}{6}\varepsilon^{\alpha\beta\gamma}\partial_\alpha X^m \partial_\beta X^n \partial_\gamma X^p \gamma_{mnp}\right)\eta = 0. \tag{9.95}$$

Here m, n, p are inner space (that is G_2-holonomy 7-manifold) indices, and γ_{mnp} are $D = 7$ Clifford algebra elements.

Spinor η in $D = 7$ can be made neither Majorana nor Weyl. But it can be chosen covariantly constant, which will mean that it transforms trivially under action of G_2 holonomy (singlet of decomposition $\mathbf{8} = \mathbf{7} + \mathbf{1}$ of spin of $Spin(7)$ - largest holonomy of $D = 7$ manifold - where $\mathbf{7}$ indicates action of $Spin(7)$ subgroup G_2, which is actual holonomy group here). With the help of such spinor one is able to construct 3-form Φ with elements $\Phi_{mnp} = \eta^T\gamma_{mnp}\eta$, which will be covariantly constant and play role of associative calibration, defined in equation BBS (9.214).

Preserved by compactification on M_7 supersymmetry transformations are performed with the help of covariantly constant on M_7 spinor η, which will signify its belonging to singlet part of G_2 holonomy. Now multiplying (9.95) by η^T from the left side, such that normalization condition $\eta^T\eta = 1$ takes place, and defining $\Phi_{mnp} = -i\eta^T\gamma_{mnp}\eta$, one gets equation

$$\partial_{[\alpha} X^m \partial_\beta X^n \partial_{\gamma]} X^p \Phi_{mnp} = \varepsilon_{\alpha\beta\gamma}.$$

The case of 4-cycle is treated in a similar manner. One first deduces equation BBS (9.220) (applied for M_7). Then one multiplies it by η^T from the left side. These will give a 4-form $\tilde{\Phi}_{mnpq} \sim \eta^T\gamma_{mnpq}\eta$, which is covariantly constant, as soon as η is. Therefore this form may be expressed as dual to associative calibration form Φ, because dualization will be provided with absolutely-anisymmetric tensor, proportional to volume form, which is covariantly constant. Contraction of covariantly constant form Φ with covariantly constant volume form will give covariantly constant form $\tilde{\Phi} \sim \star\Phi$. The rest of derivation is provided in a similar manner to that of BBS (9.219).

Problem **9.21**

Our aim is to compound independent equations which will describe constraints on parameters a_{ij} of $SO(8)$ transformations following from condition of invariancy of the form Ω, given by BBS (9.224). In this solution we describe how to find coefficient of some form dy^{ijkl} which is variation of other forms under $\delta y^i = a_{ij}y^k$. For example, pick up form 1254 (short notation for dy^{1254}). This form will be proportional to variation of some terms of form Ω, if that terms are forms, which differ from given one only by one index. Coefficient of proportionality will be appropriate sign of a_{ij}, where indices of parameter a are a couple of distinct indices of varied form and result of variation. For example, 1234 is first term of Ω, and it is to be varied: $\delta(1234) = a_{35}(1254) + \ldots$, where we omit other terms because we are looking for 1254-type form. There're 4 indices, which may differ from 1254 while other three are the same. Therefore the number of coefficient terms in front of each form will be equal to four. Our case gives

$$a_{35} + a_{64} + a_{82} - a_{71} = 0.$$

To get the number of linearly independent equations one shall find the number of ways to construct equations of this type with four terms, such that all a-parameters are different in all equations. There're 28 parameters in $SO(8)$, and therefore the number of independent equations is equal to $\frac{28}{4} = 7$, which is equal to difference in number of parameters of $SO(8)$ and $SO(7)$ (this is not a surprise, while we start with larger $SO(2k)$ group and take k-form for calibration of $SO(2k-1)$). Altogether we have the following system:

$$a_{35} + a_{64} + a_{82} - a_{71} = 0, \quad a_{36} - a_{54} + a_{72} + a_{81} = 0,$$

$$a_{37} + a_{84} - a_{62} + a_{51} = 0, \quad a_{38} - a_{74} - a_{52} - a_{61} = 0,$$

$$-a_{31} - a_{68} - a_{75} + a_{42} = 0, \quad -a_{34} - a_{78} + a_{65} - a_{21} = 0,$$

$$-a_{14} - a_{23} - a_{58} - a_{76} = 0.$$

10 Flux compactifications

`Problem` **10.1**

Actually, some calculations about transformation of spinor covariant derivative under metric conformal transformations were already performed in the solution of Problem 8.8. But we will repeat them here (in kind of different notation) for reasons pointed bellow. Local conformal transformation of metric $\hat{g}_{MN} = \Omega^2 g_{MN}$ may be reformulated as transformation of elfbein $\hat{E}^A_M = \Omega E^A_M$, for inverse elfbein this transformation looks like $\hat{E}^M_A = \Omega^{-1} E^M_A$. According to formula GSW (12.1.5) (or BBS (8.19), (8.20)) for spin connection one then has

$$\hat{\omega}^{AB}_M \Gamma_{AB} = \omega^{AB}_M \Gamma_{AB} - \Omega^{-1} E^{NA} E^A_M (\partial_N \Omega) \Gamma_{AB}.$$

According to GSW (12.1.6) (or BBS (8.18)) one then results in

$$\hat{\nabla}_M \eta = \nabla_M \eta + \frac{1}{2} \Omega^{-1} (\partial_N \Omega) \Gamma_M{}^N \eta. \tag{10.96}$$

Covariant and partial derivatives of scalar Ω are equivalent to each other.

Now we shall apply our result to derive formulae BBS (10.18). It's not straightforward for the case of first formula of BBS (10.18), because conformal rescaling factor, which is a power of Δ, depends only on internal coordinates, therefore first formula mixes indices m and μ and uses both conformal factors $\Omega_1 = \Delta^{-1/2}$ and $\Omega_2 = \Delta^{1/4}$. As for second formula, we deal only with Ω_2 and formula (10.96) gives:

$$\hat{\nabla}_m \varepsilon = \nabla_m \varepsilon + \frac{1}{8} \Delta^{-1} (\partial_n \Delta) \Gamma_m{}^n \varepsilon =$$

$$= \nabla_m \varepsilon + \frac{1}{8} \Delta^{-1} (\partial_n \Delta) (1 \otimes \gamma_m{}^n) \varepsilon.$$

Note, that for Γ-matrices one should use formulae BBS (10.8)*without* conformal factors - in old basis of elfbeins.

Now, after we've figured out this simplest example, let's consider the first case. Keeping only derivatives of Ω_i by internal coordinates in formula GSW (12.1.5), we will get

$$\Gamma_{AB}\hat{\omega}_\mu^{AB} = \Gamma_{AB}\omega_\mu^{AB} - \Omega_2^{-1}E^{nA}E_\mu^B(\partial_n\Omega_1)\Gamma_{AB} -$$

$$-\Gamma_{AB}E^{nA}\Omega_2^{-1}\left(\Omega_1^{-1}E^{\nu B}(\partial_n\Omega_1)E_{\nu C}E_\mu^C\Omega_1 + \Omega_2^{-1}E^{mB}(\partial_n\Omega_2)E_{mC}E_\mu^C\Omega_1\right),$$

where the last term vanishes, because it contains contraction $E_{mC}E_\mu^C = g_{m\mu} = 0$. Then, after simplification, one gets

$$\Gamma_{AB}\hat{\omega}_\mu^{AB} = \Gamma_{AB}\omega_\mu^{AB} - \Delta^{-\frac{7}{4}}\Delta_{,n}\Gamma_\mu{}^{\ n}$$

Using mentioned above formula

$$\nabla_\mu\varepsilon = \partial_\mu\varepsilon + \frac{1}{4}\omega_\mu^{AB}\Gamma_{AB}\varepsilon$$

for covariant derivative of spinor, and substituting expressions for factorized gamma-matrices one results in first formula BBS (10.18).

`Problem` 10.2

If $\eta = \Delta^{-1/4}\xi$, then

$$\nabla_m\eta + \frac{1}{4}\Delta^{-1}(\partial_m\Delta)\eta = \Delta^{-1/4}\nabla_m\xi,$$

and therefore BBS (10.26) gives

$$\nabla_m\xi = \Delta^{-3/4}\mathbf{F}_\mathrm{m}\xi.$$

In Majorana representation γ-matrices are real and symmetric, and spinors ξ are real. Therefore, because of due to definition BBS (10.21) one has $\mathbf{F}_\mathrm{m}^\mathrm{T} = -\mathbf{F}_\mathrm{m}$, then

$$\nabla_m\xi^\dagger = \nabla_m\xi^T = -\Delta^{-3/4}\xi^T\mathbf{F}_\mathrm{m}.$$

Therefore

$$\nabla_m J_n{}^{\ p} = \nabla_m(-i\xi^T\gamma_n{}^{\ p}\xi) = i\Delta^{-3/4}\xi^T(\mathbf{F}_\mathrm{m}\gamma_\mathrm{n}{}^{\ \mathrm{p}} - \gamma_\mathrm{n}{}^{\ \mathrm{p}}\mathbf{F}_\mathrm{m})\xi. \tag{10.97}$$

Because of internal spinors ξ are commuting and matrix

$$\mathbf{F}_\mathrm{m}\gamma_\mathrm{n}{}^{\ \mathrm{p}} - \gamma_\mathrm{n}{}^{\ \mathrm{p}}\mathbf{F}_\mathrm{m}$$

is obviously antisymmetric, then according to (10.97) one results in

$$\nabla_m J_n{}^{\ p} = 0.$$

`Problem` 10.4

In the text formulae BBS (10.23), (10.26) were derived for positive-chirality $D = 8$ spinor η: $\gamma_9\eta = \eta$. In this problem we deal with non-chiral spinor (in $D = 8$ it's Majorana spinor, which is a sum of two spinors with opposite chirality), which makes it necessary to retrieve where matrix γ_9 occurs in generalization of formula BBS (10.26). At the same time we assume that BBS (10.24)-(10.25) are valid for positive- and negative-chirality spinors separately.

Now observe that according to the first formula in the solution of Ex. 10.2, in the case of self-dual 4-form F (on internal manifold) for positive-chirality spinor η_+ one has

$$\mathbf{F}_{\mathrm{m}}\mathbf{F}^{\mathrm{m}}\eta_+ = -2\mathbf{F}^2\eta_+$$

Therefore due to BBS (10.24) one gets

$$\mathbf{F}_{\mathrm{m}}\eta_+ = 0. \tag{10.98}$$

Condition for unbroken supersymmetry for internal-indices variation is (see BBS (10.4), (10.18.2))

$$\nabla_m\varepsilon + \frac{1}{8}\Delta^{-1}(\partial_n\Delta)(1\otimes\gamma_m{}^n)\varepsilon + \frac{1}{12}\left(\Gamma_m\mathbf{F}^{(4)} - 3\mathbf{F}^{(4)}_{\mathrm{m}}\right)\varepsilon = 0.$$

In what follows we first of all employ BBS (10.14), (10.20.1), (10.20.3), (10.8) and then constraints BBS (10.24), (10.25) deduced from independent requirement of supersymmetry, then equation

$$\gamma_m{}^n = \gamma_m\gamma^n - \delta^n_m,$$

which will lead us to (as soon as in $D=3$ 2-component spinor ζ is Majoarana and considered here η is Majorana too, we don't have any complex conjugate terms in what follows)

$$0 = \zeta\otimes\left(\nabla_m\eta + \frac{1}{8}\Delta^{-1}(\partial_n\Delta)\gamma_m{}^n\eta\right) +$$

$$+\frac{1}{12}\left(\Delta^{-3/4}(1\otimes\gamma_m\mathbf{F}) + \Delta^{3/2}(1\otimes\gamma_{\mathrm{m}}\gamma_9\mathbf{f}) + 3\Delta^{3/2}\mathrm{f}_{\mathrm{m}}(1\otimes\gamma_9) - 3\Delta^{-3/4}(1\otimes\mathbf{F}_{\mathrm{m}})\right)(\zeta\otimes\eta) =$$

$$= \zeta\otimes\left(\nabla_m\eta + \left(\frac{1}{8}\Delta^{-1}(\partial_n\Delta)\gamma_m{}^n - \frac{1}{8}\Delta^{-1}\gamma_m\gamma^n\gamma_9\partial_n\Delta + \frac{3}{8}\Delta^{-1}(\partial_m\Delta)\gamma_9 - \frac{1}{4}\Delta^{-3/4}\mathbf{F}_{\mathrm{m}}\right)\eta\right) =$$

$$= \zeta\otimes\left(\nabla_m\eta - \left(\frac{1}{8}\Delta^{-1}(\partial_m\Delta)(1-\gamma_9) - \frac{1}{8}\Delta^{-1}(\partial_m\Delta)\left(3\gamma_9 - 1\right) + \frac{1}{4}\Delta^{-3/4}\mathbf{F}_{\mathrm{m}}\right)\eta\right).$$

Multiply last equation by P_- (which will give η_+ in the last term of r.h.s. of the last equation); then due to (10.98) this equation reduces to

$$\nabla_m\eta_- - \frac{3}{4}\Delta^{-1}(\partial_m\Delta)\eta_- = 0,$$

which after rescaling $\xi_- = \Delta^{-3/4}\eta_-$ may be rewritten as

$$\nabla_m\xi_- = 0.$$

Multiplication of our internal supersymmetry constraint equation by P_+ will give

$$\nabla_m\eta_+ + \frac{1}{4}\Delta^{-1}(\partial_m\Delta)\eta_+ - \frac{1}{4}\Delta^{-3/4}\mathbf{F}_{\mathrm{m}}\eta_- = 0.$$

After rescaling $\xi_+ = \Delta^{1/4}\eta_+$ this will give us

$$\nabla_m\xi_+ - \frac{1}{4}\Delta^{-3/4}\mathbf{F}_{\mathrm{m}}\xi_- = 0.$$

Problem 10.8

Cosmological constant Λ adds the term

$$S_{\Lambda} = -\int d^{10}x\sqrt{-G}\Lambda$$

to the action BBS (10.75). This term shows itself in Einstein equations as additional term in energy-momentum tensor BBS (10.83):

$$\delta_{\Lambda}T_{MN} = -\Lambda G_{MN}. \tag{10.99}$$

As one can see, transition BBS (10.84) $\to$ BBS (10.85) affects right-hand matter side by multiplication by minus factor together with some exponential $\sim e^{2A}$ factor, coming from metric ansatz BBS (10.80). In our case of additional energy-momentum tensor term (10.99) one shall add positive term $\sim \Lambda e^{2A}$ to r.h.s. of BBS (10.85). Then one proceeds in a trivial way to equation, analogous to BBS (10.86) with additional positive term on the r.h.s., which will not change therefore the line of contemplations, applied to BBS (10.86) to prove vanishing of fluxes and trivialization of warp factor.

Problem 10.9

We are on Calabi-Yau three-fold CY_3 and we want to construct $D = 4$ potential of field G_3 via method of Kaluza-Klein reduction. It means that we take G_3-term from the action BBS (10.75)

$$S = -\frac{1}{4\kappa^2}\int d^{10}x\sqrt{-G}\frac{|G_3|^2}{\mathrm{Im}\tau} \tag{10.100}$$

and perform integration over internal manifold CY_3 (which will eliminate all Kaluza-Klein excitations leaving us with only zero modes):

$$S = \int d^4x\sqrt{-g_4}\mathcal{V},$$

where scalar potential for moduli field has the form (according to problem statement complex dilation field τ is constant and may be taken out of integral):

$$\mathcal{V} = -\frac{1}{4\kappa^2\mathrm{Im}\tau}\int d^6x\sqrt{g_6}|G_3|^2.$$

Now observe that because 3-form G_3 is imaginary self-dual on 6-manifold (see BBS (10.97)), then

$$\int d^6x\sqrt{g_6}|G_3|^2 = -\int G_3\wedge\star\bar{G}_3 = -i\int G_3\wedge\bar{G}_3.$$

Therefore scalar potential is given by

$$\mathcal{V} = \frac{i}{4\kappa^2\mathrm{Im}\tau}\int G_3\wedge\bar{G}_3.$$

Then note that among harmonic 3-forms on CY_3 we have only Ω, χ_α and conjugate to them. Flux form G_3 is harmonic (on-shell) and it's also imaginary self-dual. Therefore, according to table on p. 487 BBS, we are able to expand G_3 in terms of imaginary self-dual forms χ_α and $\bar{\Omega}$:

$$G_3 = A\bar{\Omega} + B^\alpha \chi_\alpha. \tag{10.101}$$

Introduce complex-structure moduli metric

$$G_{\alpha\bar{\beta}} = -\frac{\int \chi_\alpha \wedge \bar{\chi}_{\bar{\beta}}}{\int \Omega \wedge \bar{\Omega}}.$$

Then integrating wedge product of (10.101) with $\bar{\chi}_{\bar{\beta}}$ over CY_3 one finds:

$$\int G_3 \wedge \bar{\chi}_{\bar{\beta}} = B^\alpha \int \chi_\alpha \wedge \bar{\chi}_{\bar{\beta}} = -B^\alpha G_{\alpha\bar{\beta}} \int \Omega \wedge \bar{\Omega}$$

and therefore

$$B^\alpha = -G^{\alpha\bar{\beta}} \frac{\int G_3 \wedge \bar{\chi}_{\bar{\beta}}}{\int \Omega \wedge \bar{\Omega}}.$$

Integrating wedge product of (10.101) with Ω over CY_3 gives

$$A = -\frac{\int G_3 \wedge \Omega}{\int \Omega \wedge \bar{\Omega}}.$$

Therefore we obtain the following expansion:

$$G_3 = -\frac{1}{\int \Omega \wedge \bar{\Omega}} \left(\bar{\Omega} \int G_3 \wedge \Omega + G^{\alpha\bar{\beta}} \chi_\alpha \int G_3 \wedge \bar{\chi}_{\bar{\beta}} \right).$$

We use this formula to calculate

$$\int G_3 \wedge \bar{G}_3 = \frac{1}{\left(\int \Omega \wedge \bar{\Omega}\right)^2} \left(\int G_3 \wedge \Omega \int \bar{G}_3 \wedge \bar{\Omega} \int \bar{\Omega} \wedge \Omega + G^{\alpha\bar{\beta}} G^{\bar{\gamma}\delta} \int G_3 \wedge \bar{\chi}_{\bar{\beta}} \int \bar{G}_3 \wedge \chi_\delta \int \chi_\alpha \wedge \bar{\chi}_{\bar{\gamma}} \right)$$

$$= -\frac{1}{\int \Omega \wedge \bar{\Omega}} \left(\int G_3 \wedge \Omega \int \bar{G}_3 \wedge \bar{\Omega} + G^{\alpha\bar{\beta}} \int G_3 \wedge \bar{\chi}_{\bar{\beta}} \int \bar{G}_3 \wedge \chi_\alpha \right), \tag{10.102}$$

where obviously only $A\bar{A}$- and $B\bar{B}$-types of terms have survived.

According to BBS (10.101) we know that $\int G_3 \wedge \Omega = -W$. Using BBS (9.103), (10.103), (10.105) (the later with just first equality, without equality to zero) we can also easily figure out that

$$\mathcal{D}_\alpha W = \int \chi_\alpha \wedge G_3.$$

But integration over CY_3 in r.h.s. of the last equation is impossible, because G_3 is expanded along χ_α, not $\bar{\chi}_{\bar{\alpha}}$. Therefore we add term $\int \Omega \wedge \bar{G}_3$ to superpotential W, which anyway was assumed in the text - see BBS (10.108) - and which will have no additional affection on further computations. As a result we will get

$$\mathcal{D}_\alpha W = \int \chi_\alpha \wedge \bar{G}_3.$$

This is what we now can employ in (10.102).

According to BBS (10.102), (10.103) we have Kähler metric for τ and ρ moduli space with components:

$$G_{\tau\bar{\tau}} = -\frac{1}{(\tau-\bar{\tau})^2} \to G^{\tau\bar{\tau}} = -(\tau-\bar{\tau})^2,$$

$$G_{\rho\bar{\rho}} = -\frac{3}{(\rho-\bar{\rho})^2} \to G^{\rho\bar{\rho}} = -\frac{(\rho-\bar{\rho})^2}{3}.$$

And due to BBS (10.105) (again, just formula for $\mathcal{D}_a W$, not extremal condition) we have

$$\mathcal{D}_\tau W = -\frac{W}{\tau-\bar{\tau}}, \quad \mathcal{D}_{\bar{\tau}}\bar{W} = \frac{\bar{W}}{\tau-\bar{\tau}},$$

$$\mathcal{D}_\rho W = -\frac{3W}{\rho-\bar{\rho}}, \quad \mathcal{D}_{\bar{\rho}}\bar{W} = \frac{3\bar{W}}{\rho-\bar{\rho}}.$$

Therefore we have the following identity:

$$G^{\tau\bar{\tau}}\mathcal{D}_\tau W \mathcal{D}_{\bar{\tau}}\bar{W} + G^{\rho\bar{\rho}}\mathcal{D}_\rho W \mathcal{D}_{\bar{\rho}}\bar{W} - 3|W|^2 = 4|W^2| - 3|W|^2 = |W|^2.$$

Due to BBS (10.101) this equals to

$$\int G_3 \wedge \Omega \int \bar{G}_3 \wedge \bar{\Omega}$$

which occurs in nominator of (10.102).

Collecting all results together we obtain

$$\mathcal{V} = -\frac{i}{4\kappa^2 \mathrm{Im}\tau}\frac{1}{\int \Omega\wedge\bar{\Omega}}\left(G^{a\bar{b}}\mathcal{D}_a W \mathcal{D}_{\bar{b}}\bar{W} - 3|W|^2\right),$$

where $a = \tau,\ \rho,\ \alpha$. Finally note, that from BBS (10.103) it follows that

$$e^{\mathcal{K}(\tau)} = \frac{1}{2\mathrm{Im}\tau}, \quad e^{\mathcal{K}(z^\alpha)} = \frac{1}{i\int\Omega\wedge\bar{\Omega}}.$$

Therefore

$$\mathcal{V} = \frac{1}{2\kappa^2}e^{\mathcal{K}(z^\alpha)+\mathcal{K}(\tau)}\left(G^{a\bar{b}}\mathcal{D}_a W \mathcal{D}_{\bar{b}}\bar{W} - 3|W|^2\right),$$

`Problem` **10.10**

Let's use homology $(3,1)$-cycle γ on Calabi-Yau four-fold M (assuming it's non-zero, which according to Poincare duality is valid if $F^{1,3} \neq 0$) to define moduli space coordinate $X = e^{\mathcal{K}/2}\int_\gamma \Omega$, where Ω is $(4,0)+(3,1)$ form, and Kähler potential on complex structure moduli space is given by

$$\mathcal{K} = -\ln\int\Omega\wedge\bar{\Omega}.$$

If $\tilde{\gamma}$ is corresponding $(1,3)$-cycle, then we can define coordinate $H = e^{\mathcal{K}/2}\int_{\tilde{\gamma}}\bar{\Omega}$. Define

$$Z = e^{\mathcal{K}/2}\int_\gamma \Omega = e^{\mathcal{K}/2}\int_M F\wedge\Omega,$$

where we've used Poincare duality between $\tilde{\gamma}$ and F. Using BBS (9.116) (and BBS (9.106), which excludes all extra terms from summation there) we obtain

$$Z = e^{\mathcal{K}/2}\left(\int_{\tilde{\gamma}} F \int_{\gamma} \Omega - \int_{\tilde{\gamma}} \Omega \int_{\gamma} F\right) = X.$$

One easily observes now that

$$|Z|^2 = \frac{|\int_{\gamma} \Omega|^2}{\int_M \Omega \wedge \bar{\Omega}} = |X|^2$$

is extremal at $X = 0$, where it equals to zero.

If $(1,3)$ cohomology is zero, then according to Poincare duality $(3,1)$ homology cycle γ just shrinks to zero size.

`Problem` **10.11**

From expression of Cristoffel connection through metric tensor

$$\Gamma^M_{NP} = \frac{1}{2} G^{MK}(\partial_P G_{KN} + \partial_N G_{KP} - \partial_K G_{NP})$$

and tensor transformation law for G_{MN} one may easily derive non-tensor transformation of Cristoffel onnection:

$$\Gamma^{M'}_{N'P'} = \frac{\partial x^{M'}}{\partial x^M}\frac{\partial x^N}{\partial x^{N'}}\frac{\partial x^P}{\partial x^{P'}}\Gamma^M_{NP} + \frac{\partial x^{M'}}{\partial x^S}\frac{\partial^2 x^S}{\partial x^{N'}\partial x^{P'}}. \tag{10.103}$$

Therefore torsion (see BBS (10.197))

$$T^M_{NP} = \Gamma^M_{NP} - \Gamma^M_{PN}$$

transforms as a tensor, because non-tensor part of (10.103) cancels out.

Note that according to some definition Cristoffel connection may be assumed to be symmetric in its lower indices. In that case we can construct some different affine connection, which will be equal to Cristoffel connection plus the half of the torsion.

`Problem` **10.12**

As it's suggested on BBS p. 483 to get negative term on r.h.s. of BBS (10.86) we are to add term $\sim (\alpha')^2$ of the higher order of string (α') perturbation theory to effective $D7$-brane action BBS (10.90). This term occurs to be equal to

$$\delta S_{loc} = -(\alpha')^2 T_7 \int_{R^4\times\Sigma} C_4 \wedge \frac{\pi^4 p_1(R)}{3}, \tag{10.104}$$

where C_4 is type-IIB gauge field. Due to pointed out on pp. 233-234 BBS or Polchinski 8.7 we have the relation $T_p = \frac{T_{p-1}}{2\pi\sqrt{\alpha'}}$ between tension of Dp-branes. According to BPS condition $\mu_p \sim T_p$ which for $p = 3$ in Einstein frame is $\mu_3 = T_3$, we then have $\mu_3 = (2\pi)^4(\alpha')^2 T_7$, and therefore we can re-express our correction to local source action as BBS (10.91). Then we substitute expression BBS (10.51) for first Pontryagin class into (10.104), which will give

$$\delta S_{loc} = \frac{T_7}{96}(2\pi\alpha')^2 \int_{R^4\times\Sigma} C_4 \wedge \mathrm{tr}(R^2).$$

Corresponding correction to r.h.s. of equation BBS (10.86) is given by BBS (10.87) with account to BBS (10.88), (10.89). Then note that

$$\mathrm{tr}(R^2) = d\omega_3,$$

where $\omega_3 = tr(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega)$ is Chern-Simons 3-form, where ω is spin connection gauge form (see e.g. solution of Problem 5.9 here), and therefore the term

$$d\omega_3 = \frac{1}{4!}(d\omega_3)_{mnpq}(dx^m \wedge dx^n \wedge dx^p \wedge dx^q).$$

won't give contribution to variation by metric tensor. We then have

$$C_4 \wedge \mathrm{tr}R^2 = \frac{1}{4!}(d\omega_3)C_{\mu\nu\lambda\rho}dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho,$$

where due to ansatz BBS (10.80) mixed components of $\mathrm{tr}(R^2)$ and C_4 obviously vanish and due to BBS (10.81), following from $D = 4$ space-time manifold being Poincare invariant, components of 4-form C_4 along internal manifold vanish. Symmetry considerations also require $C_{\mu\nu\lambda\rho} = C(y)E_{\mu\nu\lambda\rho} = \sqrt{-g_4}C(y)\varepsilon_{\mu\nu\lambda\rho}$ for components of C_4, where $C(y)$ is a scalar on internal manifold ($y \sim x^m$), which we set $C(y) = 1$ for further convenience. We then have

$$C_4 \wedge \mathrm{tr}R^2 = \sqrt{-g_4}d^4x d\omega_3.$$

Due to pointed here in variation by external metric takes part only

$$-\frac{2}{\sqrt{-g_4}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g_4} = g_{\mu\nu},$$

while variation by internal metric vanishes. As a result energy-momentum tensor is given by

$$T_{\mu\nu} = \frac{T_7}{96}(2\pi\alpha')^2 g_{\mu\nu}\int_\Sigma d\omega_3,$$

therefore due to BBS (10.88) one has

$$\mathcal{J}_{loc} = -\frac{T_7}{96}(2\pi\alpha')^2\int_\Sigma d\omega_3.$$

`Problem` **10.13**

According to BBS (10.119), (10.124) metric on conifold is given by

$$ds^2 = dr^2 + \frac{r^2}{9}(g^5)^2 + \frac{r^2}{6}\sum_{i=1}^{4}(g^i)^2.$$

Then we know that conifold is a complex manifold, because it was originally presented as hypersurface in C^4 by equation BBS (10.117). Conifold describes a conical singularity parts of Calabi-Yau three-folds, therefore conifold is Kähler. According to BBS (9.269) we can introduce

Kähler form on it. First we are to determine metric components in complex coordinates. Let's define complex coordinates $(z_1,\ z_2,\ z_3)$ by the following formulae:

$$dz_1 = dr + \frac{i}{3} r g^5,$$

$$dz_2 = g^2 + i g^3,$$
$$dz_3 = g^4 + i g^1.$$

Then metric may be rewritten as

$$ds^2 = dz_1 d\bar{z}_1 + \frac{r^2}{6} dz_2 d\bar{z}_2 + \frac{r^2}{6} dz_3 d\bar{z}_3,$$

the non-zero metric components are $g_{1\bar{1}} = \frac{1}{2},\ g_{2\bar{2}} = \frac{r^2}{12},\ g_{3\bar{3}} = \frac{r^2}{12}$ and complex conjugate to them (see BBS (9.266)). Kähler form elements are given by $J_{a\bar{b}} = i g_{a\bar{b}}$, and complex conjugate to them $J_{\bar{a}b} = -i g_{\bar{a}b} = -i (g_{a\bar{b}})^\star$. Kähler form is then given by

$$J = i g_{a\bar{b}} dz^a \wedge d\bar{z}^b = \frac{i}{2} \left(dz_1 \wedge d\bar{z}_1 + \frac{r^2}{6} dz_2 \wedge d\bar{z}_2 + \frac{r^2}{6} dz_3 \wedge d\bar{z}_3 \right).$$

Substitute formulae for z_i:

$$J = \frac{2r}{3} dr \wedge g^5 + \frac{r^2}{3} \left(g^2 \wedge g^3 + g^4 \wedge g^1 \right). \tag{10.105}$$

Using formulae BBS (10.122), formula (10.105) may be rewritten as

$$J = \frac{2r}{3} dr \wedge g^5 + \frac{r^2}{3} (e^2 \wedge e^1 + e^3 \wedge e^4).$$

Then note that formulae BBS (10.133), (10.135) make G_3 imaginary self-dual. Therefore primitivity condition $\star G_3 \wedge J = 0$ may be checked as $G_3 \wedge J = 0$. Let's aim to rewrite ω_2 in terms of $dr,\ g^i$ forms, to get G_3 expressed through them too. To do it one have to use BBS (10.122), (10.123) (10.134), which gives $\omega_2 = \frac{1}{2} \left(g^1 \wedge g^2 + g^3 \wedge g^4 \right)$. Therefore (we use BBS (10.77) with $C_0 = 0$ and $g_s = e^\Phi$, as it's assumed on pp. 491-492 BBS)

$$G_3 = \frac{M\alpha'}{4} \left(g^5 - \frac{3i}{r} dr \right) \wedge \left(g^1 \wedge g^2 + g^3 \wedge g^4 \right).$$

Now it's obvious task to check primitivity of G_3: each term of wedge product $G_3 \wedge J$ contains some basic 1-form twice.

`Problem` **10.14**

We are to find a for which equality

$$\star H = -e^{-a\Phi} d(e^{a\Phi} J)$$

holds.

From BBS (10.220) we know that H is $(2,1)+(1,2)$-form, expressed through fundamental form according to formula $H_{ab\bar{c}} = 2i\partial_{[a}J_{b]\bar{c}}$. Here factor 2 arose from antisymmetrization in $\partial_{[m}J_{b]\bar{c}}$, which replaced that tensor without any antisymmetrization, as it naturally arises from BBS (10.220). We perform antisymmetrization only among either holomorphic or antiholomorphic indices, but not between them, and insert factors $\frac{1}{p!}$ corresponding to antisymmetrization of p indices. Square brackets assume antisymmetrization, which also assumes corresponding factor $\frac{1}{p!}$. Finally, all $(2,1)$ components $H_{ab\bar{c}}$, $H_{a\bar{c}b}$, $H_{\bar{c}ab}$ make the same impact on H, and therefore expansion of H in basis of 3-forms $dz^a \wedge dz^b \wedge d\bar{z}^{\bar{c}}$ contains $H_{ab\bar{c}}$ 3 times, if other 2 terms are omitted. We assume that antisymmetrization (which is that between holomorphic and antiholomorphic indices) *was* already performed, and corresponding factor $\frac{1}{3}$ canceled factor 3. Therefore

$$H = \frac{1}{2}H_{ab\bar{c}}dz^a \wedge dz^b \wedge d\bar{z}^{\bar{c}} + c.c.$$

If one substitutes now $H_{ab\bar{c}} = 2i\partial_{[a}J_{b]\bar{c}}$ mentioned above, one will get first term of BBS (10.220) (where differentiation doesn't bring 2!-related factor, as it's also pointed above). We will not write *c.c.* in what follows for shortness.

Hodge dual form is given by

$$\star H = \frac{1}{2}H_{ab\bar{c}} \star (dz^a \wedge dz^b \wedge d\bar{z}^{\bar{c}}) =$$

$$= \frac{1}{2}H_{ab\bar{c}}\frac{1}{2}E^{ab\bar{c}d\bar{e}\bar{f}}g_{d\bar{k}}g_{m\bar{e}}g_{n\bar{f}}dz^m \wedge dz^n \wedge d\bar{z}^{\bar{k}}.$$

All indices are complex: holomorphic or antiholomorphic. Antisymmetric tensor equals to

$$E^{ab\bar{c}d\bar{e}\bar{f}} = -E^{abd\bar{c}\bar{e}\bar{f}} = i(g^{a\bar{c}}g^{b\bar{e}}g^{d\bar{f}} \pm \text{permutations}),$$

where we fix, e.g., all holomorphic indices and permutate antiholomorphic. Then one can easily figure out that

$$(g^{a\bar{c}}g^{b\bar{e}}g^{d\bar{f}} \pm \text{permutations})g_{d\bar{k}}g_{m\bar{e}}g_{n\bar{f}} =$$
$$= 2(2g^{\bar{c}[a}\delta^{b]}_{[m}g_{n]\bar{k}} + \delta^{\bar{c}}_{\bar{k}}\delta^{[a}_{m}\delta^{b]}_{n}).$$

We can use it to proceed with our calculation of $\star H$. Inserting also expression of H-form components through fundamental form, pointed above, we will get:

$$\star H = -\frac{1}{2}\partial_{[a}J_{b]\bar{c}}(2g^{\bar{c}[a}\delta^{b]}_{[m}g_{n]\bar{k}} + \delta^{\bar{c}}_{\bar{k}}\delta^{[a}_{m}\delta^{b]}_{n})dz^m \wedge dz^n \wedge d\bar{z}^{\bar{k}}.$$

We can't use metric tensor to rise and lower indices under the sign of partial derivative, but we can take that indices out of sign of partial derivative using antisymmetrization property. That indeed occurs in all cases of our interest:

$$\star H = -\frac{1}{2}\left(2g_{n\bar{k}}\partial^{[\bar{c}}J_{m]\bar{c}} + \partial_{[m}J_{n]\bar{k}}\right)dz^m \wedge dz^n \wedge d\bar{z}^{\bar{k}} =$$

$$= g_{m\bar{k}}\partial^{[\bar{c}}J_{n]\bar{c}}dz^m \wedge dz^n \wedge d\bar{z}^{\bar{k}} - \frac{1}{2}\partial_{[m}J_{n]\bar{k}}dz^m \wedge dz^n \wedge d\bar{z}^{\bar{k}} =$$

$$= g_{m\bar{k}}\partial^{[\bar{c}}J_{n]\bar{c}}dz^m \wedge dz^n \wedge d\bar{z}^{\bar{k}} - dJ.$$

From the other side

$$-e^{-a\Phi}d(e^{a\Phi}J) = -ad\Phi \wedge J - dJ.$$

According to BBS (10.221)

$$d\Phi = -\frac{1}{2}g^{b\bar{c}}H_{mb\bar{c}}dz^m = -ig^{b\bar{c}}\partial_{[m}J_{b]\bar{c}}dz^m.$$

Therefore

$$d\Phi \wedge J = -\frac{i}{2}\partial_{[m}J_{b]\bar{c}}J_{n\bar{k}}g^{b\bar{c}}dz^m \wedge dz^n \wedge d\bar{z}^{\bar{k}} =$$

$$= \frac{i}{2}ig_{n\bar{k}}g^{b\bar{c}}\partial_{[b}J_{m]\bar{c}}dz^m \wedge dz^n \wedge d\bar{z}^{\bar{k}} =$$

$$= \frac{1}{2}g_{m\bar{k}}\partial^{[\bar{c}}J_{n]\bar{c}}dz^m \wedge dz^n \wedge d\bar{z}^{\bar{k}}.$$

Now one can clearly see that it should be $a = -2$.

`Problem` **10.15**

In this problem we are to verify formulae BBS (10.170), (10.173), (10.176) and (10.177). First two formulae are trivial consequence of things pointed on p. 500 BBS, while the last two formulae may be obtained in a way similar to that used in the solution of Problem 10.9.

Anyway, on page 469 BBS it was pointed out an important alteration property of sign of Hodge dual of the form from Lefschetz decomposition. In application to Lefschetz decomposition of $F^{2,2}$, which is provided by BBS (10.169), it means that first and third forms in r.h.s. of BBS (10.169) are self-dual while second is anti self-dual. Therefore

$$\star F^{2,2} = F_0^{2,2} - J \wedge F_0^{1,1} + J \wedge J \wedge F_0^{0,0},$$

which obviously assumes BBS (10.170). Using the result of Ex. 10.4 one gets in a similar way BBS (10.172). Then due to obvious even-rank form property $F^{3,1} \wedge F^{1,3} = F^{1,3} \wedge F^{3,1}$ one immediately gets BBS (10.173).

Introduce Kähler potential BBS (10.64)

$$\mathcal{K}^{3,1} = -\log\left(\int \Omega \wedge \bar{\Omega}\right) \tag{10.106}$$

and corresponding metric

$$G_{I\bar{J}} = -\frac{\int \chi_I \wedge \bar{\chi}_{\bar{J}}}{\int \Omega \wedge \bar{\Omega}}$$

on complex structure moduli space. Note, that Kähler potential doesn't contain factor i inside logarithm, which was the case when we were dealing with 3-form Ω on CY_3 to achieve reality of Kähler potential and hermicity of Kähler metric. Now $\Omega \wedge \bar{\Omega} = \bar{\Omega} \wedge \Omega$ and we don't need any i.

Suppose we have $h^{3,1}$ basic χ_I $(3,1)$-forms. Then we can make an expansion

$$F^{3,1} = A^I \chi_I.$$

Making wedge product of both sides of the last equality with $\bar{\chi}_{\bar{J}}$ after integration the result over the whole CY_4 and taking into account formula for Kähler moduli space metric one gets

$$F^{3,1} = -G^{I\bar{J}}\chi_I \frac{\int F^{3,1} \wedge \bar{\chi}_{\bar{J}}}{\int \Omega \wedge \bar{\Omega}}.$$

In a similar way one easily gets (again, remember that 4-forms wedge-commute)

$$F^{1,3} = -G^{K\bar{L}}\bar{\chi}_{\bar{L}} \frac{\int F^{1,3} \wedge \chi_K}{\int \Omega \wedge \bar{\Omega}}.$$

Then we can calculate the value of

$$\int F^{3,1} \wedge F^{1,3} = -\frac{1}{\int \Omega \wedge \bar{\Omega}} G^{I\bar{J}} \int F^{3,1} \wedge \bar{\chi}_{\bar{J}} \int F^{1,3} \wedge \chi_I.$$

According to the moduli sense of Kähler covariant derivative, which is shown by BBS (9.122), one has $\mathcal{D}_I\Omega = \chi_I$. Therefore due to expression for superpotential (appropriately normalized)

$$W^{1,3} = \int \Omega \wedge F^{1,3},$$

one gets:

$$\int F^{3,1} \wedge F^{1,3} = -\frac{1}{\int \Omega \wedge \bar{\Omega}} G^{I\bar{J}} \mathcal{D}_I W^{1,3} \mathcal{D}_{\bar{J}} \bar{W}^{1,3}.$$

From equation (10.106) one gets expression $\int \Omega \wedge \bar{\Omega} = e^{-\mathcal{K}^{3,1}}$, therefore

$$\int F^{3,1} \wedge F^{1,3} = -e^{\mathcal{K}^{3,1}} G^{I\bar{J}} \mathcal{D}_I W^{1,3} \mathcal{D}_{\bar{J}} \bar{W}^{1,3}.$$

`Problem` **10.17**

From BBS (10.247) it follows gravitational part of low-energy effective action of the form

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{|g|} R, \tag{10.107}$$

which in the standard form with Newton constant after compactification to four dimensions has the view

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{|g|} R. \tag{10.108}$$

Simple reduction of action (10.107) to four dimensions gives additional multiplier $(\pi d)\mathcal{V}$ for four-dimensional action. Comparing to (10.108) implies

$$G_4 = \frac{\kappa_{11}^2}{8\pi^2 \mathcal{V} d}.$$

In a similar way, when we have to perform only $\mathcal{V}$-reduction of ten-dimensional theory on the boundary, we can compare gauge actions: from BBS (10.247) it follows

$$S = -\frac{1}{8\pi(4\pi\kappa_{11}^2)^{2/3}} \int d^{10}x \sqrt{|g|} |F|^2 \rightarrow$$

$$\to -\frac{\mathcal{V}}{8\pi(4\pi\kappa_{11}^2)^{2/3}}\int d^4x\sqrt{|g|}|F|^2;$$

and it's to be compared to

$$S = -\frac{1}{16\pi\alpha_U}\int d^4x\sqrt{|g|}|F|^2.$$

The result is

$$\alpha_U = \frac{(4\pi\kappa_{11}^2)^{2/3}}{2\mathcal{V}}.$$

`Problem` **10.18**

Vanishing of supersymmetry variation of dilatino requires

$$-\frac{1}{2}(\partial\!\!\!/\ \Phi)\varepsilon + \frac{1}{4}\mathbf{H}\varepsilon = 0.$$

We have decomposition of chiral spinor parameter of susy transformation:

$$\varepsilon = \zeta_+ \otimes \eta_+ + \zeta_- \otimes \eta_-.$$

Non-zero flux components and dilaton derivatives are given by BBS (10.202). Then according to Γ-matrices decomposition BBS (10.209) one gets

$$(\partial\!\!\!/\ \Phi)\varepsilon = (\partial_m\Phi)(\gamma_5\otimes\gamma^m)(\zeta_+\otimes\eta_+ + \zeta_-\otimes\eta_-) =$$
$$= (\partial_m\Phi)(\zeta_+\otimes(\gamma^m\eta_+) - \zeta_-\otimes(\gamma^m\eta_-)) =$$
$$= (\partial_a\Phi)(\zeta_+\otimes(\gamma^a\eta_+)) - (\partial_{\bar a}\Phi)(\zeta_-\otimes(\gamma^{\bar a}\eta_-)),$$

where in the last line we used chirality condition for spinors on internal manifold (pointed, e.g., on the top of p. 514 BBS);
and

$$\mathbf{H}\varepsilon = \frac{1}{3!}\mathrm{H}_{\mathrm{MNP}}\Gamma^{\mathrm{MNP}}\varepsilon = \frac{1}{3!}\mathrm{H}_{\mathrm{mnp}}\Gamma^{\mathrm{mnp}}\varepsilon =$$
$$= \frac{1}{3!}H_{mnp}(\gamma_5\otimes\gamma^{mnp})(\zeta_+\otimes\eta_+ + \zeta_-\otimes\eta_-) =$$
$$= \frac{1}{3!}(3H_{\bar a bc}(\zeta_+\otimes(\gamma^{\bar a bc}\eta_+)) - 3H_{a\bar b\bar c}(\zeta_-\otimes(\gamma^{a\bar b\bar c}\eta_-)),$$

where in the last line we've used the fact that according to BBS (10.220) 3-form H has complex structure $(2,1)+(1,2)$. Then summarizing some results we get

$$-2(\partial_b\Phi)(\gamma^b\eta_+) + \frac{1}{2}H_{\bar a bc}\gamma^{\bar a bc}\eta_+ = 0$$

and complex conjugate to it.

Finally, as soon as for chiral spinor on internal manifold we have $\gamma^{\bar a}\eta_+ = 0$, then we obtain

$$\gamma^{\bar a bc}\eta_+ = 2(g^{\bar a b}\gamma^c - g^{\bar a c}\gamma^b)\eta_+.$$

Using this result one finally gets

$$\partial_a\Phi = -\frac{1}{2}H_{ab\bar c}g^{b\bar c}.$$

Problem 10.19

This is just a routine. We will follow the line of calculations in S. Weinberg "Cosmology", mostly paragraph 1.5.

Einstein equations look like

$$R_{\mu\nu} = -\frac{1}{M_P^2} S_{\mu\nu},$$

where reduced Planck mass is defined as $M_P = (8\pi G)^{-1/2}$. Ricci tensor is given by

$$R_{\mu\nu} = \partial_\nu \Gamma^\lambda_{\lambda\mu} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma},$$

and it was introduced tensor, constructed out of energy-momentum tensor:

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda_\lambda.$$

Cristoffel connection is given by

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}).$$

For FRW ansatz BBS (10.265) it obviously means

$$\Gamma^\mu_{00} = \Gamma^0_{00} = 0,$$

which in a less trivial but also simple way may be accompanied with elements

$$\Gamma^0_{ij} = a\dot{a}\tilde{g}_{ij},$$

$$\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j,$$

$$\Gamma^i_{jl} = \tilde{\Gamma}^i_{jl} = k\tilde{g}_{jl}x^i, \tag{10.109}$$

where tilde denotes purely spatial values without $a(t)$ factor; spatial metric is given by

$$\tilde{g}_{ij} = \delta_{ij} + k\frac{x^i x^j}{1 - k\mathbf{x}^2},$$

where x^i are 3 flat coordinates of $D = 4$ (pseudo-)Euclidean flat space, in which $D = 3$ spatial subspace of $D = 4$ FRW space-time is embedded as hypersurface. If it's a flat case ($k = 0$), then x^i are just coordinates of $D = 3$ Euclidean space. In the case of hypersphere ($k = -1$) x^i are 'spatial' coordinates of $D = 4$ pseudo-Euclidean space with line element $ds^2 = dz'^2 - d\mathbf{x}'^2$, where $z'^2 - \mathbf{x}'^2 = \mathrm{a}^2$ gives our spatial part of FRW space, and $x^i = \frac{x'^i}{a}$ are dimensionless coordinates used in notation above. Finally in spherical case we deal with $D = 4$ Euclidean space and study hypersurface $z'^2 + \mathbf{x}'^2 = \mathrm{a}^2$. Also note that δ_{ij} is metric tensor of Euclidean space (in flat coordinates). Then we can calculate connection-related elements:

$$\partial_t \Gamma^0_{ij} = \tilde{g}_{ij}\frac{d}{dt}(a\dot{a}), \quad \partial_t \Gamma^i_{i0} = 3\frac{d}{dt}\left(\frac{\dot{a}}{a}\right), \tag{10.110}$$

and

$$\Gamma^0_{ik}\Gamma^k_{j0} = \tilde{g}_{ij}\dot{a}^2, \quad \Gamma^0_{ij}\Gamma^l_{0l} = 3\tilde{g}_{ij}\dot{a}^2, \quad \Gamma^i_{0j}\Gamma^j_{i0} = 3\left(\frac{\dot{a}}{a}\right)^2, \tag{10.111}$$

and Ricci tensor components

$$R_{ij} = \tilde{R}_{ij} - \partial_t \Gamma^0_{ij} + \Gamma^0_{ik}\Gamma^k_{j0} + \Gamma^k_{i0}\Gamma^0_{jk} - \Gamma^0_{ij}\Gamma^l_{0l},$$

$$R_{00} = \partial_t \Gamma^i_{i0} + \Gamma^i_{0j}\Gamma^j_{0i},$$

which according to (10.110) and (10.111) may be rewritten as

$$R_{ij} = \tilde{R}_{ij} - 2\dot{a}^2 \tilde{g}_{ij} - a\ddot{a}\tilde{g}_{ij}, \tag{10.112}$$

$$R_{00} = 3\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) + 3\left(\frac{\dot{a}}{a}\right)^2 = 3\frac{\ddot{a}}{a}.$$

Using (10.109) we can calculate $\tilde{R}_{ij}$ at $\mathbf{x} = 0$:

$$\tilde{R}_{ij} = \partial_j \Gamma^l_{li} - \partial_l \Gamma^l_{ji} = -2k\delta_{ij}.$$

At $\mathbf{x} = 0$ it takes place $\delta_{ij} = g_{ij}$, therefore because of $\mathbf{x}$ is a scalar, we can apply our result to all points, not just $\mathbf{x} = 0$:

$$\tilde{R}_{ij} = -2k\tilde{g}_{ij}.$$

Then from (10.112) we proceed to

$$R_{ij} = -(2k + 2\dot{a}^2 + a\ddot{a})\tilde{g}_{ij}.$$

To compute $S_{\mu\nu}$ first of all note that energy-momentum tensor components are

$$T_{00} = \rho, \quad T_{i0} = 0, \quad T_{ij} = pg_{ij} = pa^2\tilde{g}_{ij},$$

and therefore

$$S_{ij} = \frac{1}{2}(\rho - p)a^2\tilde{g}_{ij},$$

$$S_{00} = \frac{1}{2}(\rho + 3p).$$

Therefore Einstein equations give us

$$\frac{2k}{a^2} + \frac{2\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} = \frac{1}{2M_P^2}(\rho - p).$$

$$\frac{3\ddot{a}}{a} = -\frac{1}{2M_P^2}(3p + \rho).$$

Subtracting three times first equation and the second equation one gets BBS (10.268), while second equation corresponds to BBS (10.269), where cosmological constant may be retrieved as additional term to ρ and p with $\rho = \Lambda M_P^2$ and $p = -\rho$.

`Problem` **10.20**

Suppose that the following conditions take place:

$$\dot{\phi}^2 \ll V(\phi), \tag{10.113}$$

$$\ddot{\phi} \ll H\dot{\phi}, \tag{10.114}$$

and therefore equations BBS (10.287) and BBS (10.288) are valid as slow-roll approximation to equations BBS (10.282), (10.283). Let's use (10.114) to be able to employ BBS (10.288) approximation to conclude that $\dot{\phi} \sim -\frac{V'}{H}$. Then from (10.113) it follows that

$$\frac{(V')^2}{H^2} \ll V. \tag{10.115}$$

Also from (10.113) it follows that we can employ BBS (10.287) approximation, which gives $H^2 \sim \frac{V}{M_P^2}$. From (10.115) we then obtain

$$\left(\frac{V'}{V}\right)^2 M_P^2 \ll 1.$$

Therefore BBS (10.289) is necessary consequence of slow-roll approximation BBS (10.287), (10.288).

If again we consider slow-roll approximation conditions (10.113), (10.114), then we can use approximate equations BBS (10.287), (10.288). Divide BBS (10.287) by square of BBS (10.288):

$$\frac{V}{M_P^2}\dot{\phi}^2 \sim V'^2.$$

Differentiation by ϕ will evidently give

$$\frac{\dot{\phi}^2}{M_P^2} \sim V''.$$

Therefore

$$M_P^2|V''/V| \sim |\dot{\phi}^2/V| \ll 1,$$

which is condition BBS (10.290).

As a result both BBS (10.289), (10.290) are necessary conditions for slow-roll approximation.

From the other side BBS (10.289), (10.290) alone are incapable to give slow-roll approximation conditions (10.113), (10.114), because they depict necessary conditions on potential V but do not give relations between this potential and derivatives of inflaton field ϕ. But if we hold approximation necessary condition BBS (10.290) and approximate equation BBS (10.288), we will be able to get another approximate equation BBS (10.287). Indeed, first of all note that differentiation of BBS (10.282), (10.283) gives

$$2HH' = \frac{V'}{3M_P^2},$$

$$3H'\dot{\phi} = -V'',$$

from which it obviously follows that

$$\frac{\dot{\phi}}{H} = -2M_P^2\frac{V''}{V'}. \tag{10.116}$$

Formula BBS (10.288) gives $V' \sim -3H\dot{\phi}$, and therefore from (10.116) we proceed to

$$\frac{\dot{\phi}}{H} \sim M_P^2 \frac{V''}{H\dot{\phi}},$$

which evidently gives

$$\dot{\phi}^2 \sim M_P^2 |V''|.$$

From BBS (10.290) it then obviously follows (10.113) and therefore approximate equation BBS (10.287).

11 Black holes in string theory

Problem 11.1

Geodesic equation for massive particle with proper time τ looks like

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0.$$

In the non-relativistic case $|\mathbf{v}| \ll 1$ we have $d\tau^2 = dt^2(1-v^2) \simeq dt^2$ and relation between components of 4-velocity vector $u^0 = \frac{dx^0}{d\tau} \simeq 1 \gg \mathbf{u} \simeq |\mathbf{v}| \simeq |\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}}|$. Therefore we proceed to equations of motion

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{00} = 0, \tag{11.117}$$

where latin indices stand for three spatial components. In the weak gravitational field we can expand metric tensor as small deviation from flat metric: $g_{\mu\nu} = \eta_{\mu\nu} + \tilde{g}_{\mu\nu}$ (with $|\tilde{g}_{\mu\nu}| \ll 1$), and therefore Cristoffel connections are expressed as

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}\eta^{\mu\rho}(\tilde{g}_{\rho\nu,\lambda} + \tilde{g}_{\rho\lambda,\nu} - \tilde{g}_{\nu\lambda,\rho}),$$

where once we use non-vanishing derivatives of metric, which are associated with $\tilde{g}_{\mu\nu}$ anyway, we are to use in all other multipliers flat metric $\eta_{\mu\nu}$. As a result

$$\Gamma^i_{00} = -\Gamma_{i,00} = -\frac{1}{2}(2\tilde{g}_{i0,0} - \tilde{g}_{00,i}).$$

In the case of static metric then we have

$$\Gamma^i_{00} = \frac{1}{2}\tilde{g}_{00,i}.$$

Then from (11.117) we proceed to

$$\frac{d^2x^i}{dt^2} = -\frac{1}{2}\tilde{g}_{00,i}.$$

In the classical mechanics we would have $-\phi_{,i}$ on the r.h.s. of the last equation. Therefore we conclude that

$$\phi = \phi_0 - \frac{1}{2}\tilde{g}_{00},$$

or more concretely

$$\phi = \phi_0 - \frac{G_4 M}{r}.$$

As soon as $\phi(r = \infty) = 0$, then

$$\phi = -\frac{G_4 M}{r},$$

which is an expression for Newtonian potential Φ. Therefore we conclude that Newtonian potential indeed relates to metric as $\Phi = -\frac{\tilde{g}_{00}}{2}$.

Problem 11.4

Scwarzschild metric in D dimensions is given by formulae BBS (11.9)-(11.11):

$$ds^2 = -\left(1 - \left(\frac{r_H}{r}\right)^{D-3}\right) dt^2 + \left(1 - \left(\frac{r_H}{r}\right)^{D-3}\right)^{-1} dr^2 + r^2 d\Omega_{D-2}^2. \tag{11.118}$$

When interval is written in terms of dr, dt differentials, or metric tensor is written in t, r coordinates, then we have coordinate singularity at the horizon $r = r_H$. To get non-singular at the horizon metric let's look for the following form of interval (without S^2 part):

$$ds^2 = -F(R)\left[R^2 \frac{dt^2}{4r_H^2} - dR^2\right].$$

Comparing to (11.118) one concludes that

$$F(R)R^2 = 4r_H^2 \left(1 - \left(\frac{r_H}{r}\right)^{D-3}\right), \tag{11.119}$$

$$F(R)dR^2 = \left(1 - \left(\frac{r_H}{r}\right)^{D-3}\right)^{-1} dr^2. \tag{11.120}$$

Divide (11.120) by (11.118) and take square root, assuming that new radial coordinate R takes only non-negative values:

$$2\frac{dR}{R} = \frac{d(r/r_H)}{1 - \left(\frac{r_H}{r}\right)^{D-3}}.$$

Integration will give us

$$R = R_0 \exp\left(\frac{1}{2}\int \frac{d(r/r_H)}{1 - \left(\frac{r_H}{r}\right)^{D-3}}\right),$$

where R_0 is a constant of integration. Therefore one concludes that

$$F = \left(\frac{2r_H}{R_0}\right)^2 \left(1 - \left(\frac{r_H}{r}\right)^{D-3}\right) \exp\left(-\int \frac{d(r/r_H)}{1 - \left(\frac{r_H}{r}\right)^{D-3}}\right).$$

Here one has value $1 - \left(\frac{r_H}{r}\right)^{D-3}$, which is zero at the horizon, and exponenta of negative value, because integral is taken of positive function (when $r \geq r_H$) and there's a minus sign in front of integral. Therefore nothing will diverge at the horizon, that is $F(R(r))$ is non-singular at $R = R(r_H)$.

One can then define coordinates

$$V = Re^{\frac{t}{2r_H}}, \quad U = -Re^{-\frac{t}{2r_H}},$$

in terms of which interval takes the form

$$ds^2 = F(R)dUdV,$$

and Kruskal-Szekeres coordinates

$$u = \frac{1}{2}(U - V), \quad v = \frac{1}{2}(U + V),$$

in terms of which interval is written as

$$ds^2 = F(R)(dv^2 - du^2).$$

`Problem` **11.5**
Consider $(t,\ r)$ part of non-extremal RN black hole interval BBS (11.60):

$$ds^2 = -h\lambda^{-2/3}dt^2 + h^{-1}\lambda^{1/3}dr^2.$$

Expand radial coordinate near horizon r_0:

$$r = r_0(1 + \rho^2).$$

In what follows we do not keep vanishing powers of ρ higher than 2. Therefore

$$h \simeq 2\rho^2,$$

$$\lambda \simeq \prod_{i=1}^{3} \left[\cosh^2 \alpha_i - 2\rho^2 \sinh^2 \alpha_i\right],$$

and therefore

$$h\lambda^{-2/3} \simeq 2\rho^2 \prod_{i=1}^{3} \cosh^{-\frac{4}{3}} \alpha_i,$$

$$h^{-1}\lambda^{1/3} = \frac{1}{2}\rho^{-2} \prod_{i=1}^{3} \cosh^{\frac{2}{3}} \alpha_i.$$

Using expressions BBS (11.64), (11.65) one will get

$$ds^2 \simeq 2\rho^2 \left(\frac{r_0}{r_H}\right)^4 d\tau^2 + 2r_H^2 d\rho^2 \simeq$$

$$\simeq 2r_H^2 \left(d\rho^2 + \rho^2 \left(\frac{d\tau}{r_H^3/r_0^2}\right)^2\right),$$

from which follows the expression for inverse temperature:

$$\beta = 2\pi \frac{r_H^3}{r_0^2} = 2\pi r_0 \prod_{i=1}^{3} \cosh \alpha_i.$$

When $r_0 \to 0$ in extremal RN black hole limit, then $r_i = r_0 \sinh \alpha_i$ remain unchanged, then obviously $\beta \to \infty$ and temperature tends to zero.

`Problem` **11.6**

From formula BBS (8.22), which is proved in the solution of Problem 8.12, we conclude about tension of $M2$-brane:

$$T_{M2} = \frac{2\pi}{(2\pi \ell_s)^3 g_s}.$$

We consider three $M2$-branes wrapping compact subspace $T^6 = T^2 \times T^2 \times T^2$ of $D = 11$ space-time of M-theory. We define the volume (surface area) of each T^2 to be $(2\pi)^2 V$. Then Newton constant in $D = 5$ is connected to that in $D = 11$ by formula

$$G_5 = \frac{G_{11}}{(2\pi)^6 V^3},$$

and therefore according to formula BBS (8.9) for G_{11} (where $\ell_P = \ell_s g_s^{\frac{1}{3}}$) one has

$$G_5 = \frac{\pi g_s^3 \ell_s^9}{4V^3}.$$

Entropy of black hole is given by BH formula

$$S = \frac{A}{4G_5},$$

where area of horizon surface is determined by BBS (11.48)

$$A = 2\pi^2 r_1 r_2 r_3,$$

with parameters $r_i = 2\sqrt{\frac{G_5 M_i}{\pi}}$, as it's stated in BBS (11.49), where $M_i = Q_i T_{M2}$. Collecting all facts together, we can easily figure out that

$$S = 4\sqrt{\pi G_5 M_1 M_2 M_3} = 2\pi \sqrt{Q_1 Q_2 Q_3},$$

which coincides with general symmetric result BBS (11.56) with account to BBS (11.59).

`Problem` **11.7**

What we know is that real-valued z_i given by BBS (11.89) give $\Delta = Q_1 Q_2 Q_3 Q_4$ from equation BBS (11.88). Here we perform transformation (and complexification):

$$Q_1 \to Q_1 + iP_1,$$

which means

$$z_i \to z_i + \frac{i}{4} P_1.$$

Using this formula one can easily figure out the following substitution rules:

$$\sum |z_i|^4 \to \sum |z_i|^4 + \frac{1}{8} P_1^2 \sum |z_i|^2 + \frac{1}{64} P_1^4,$$

$$\left(\sum |z_i|^2\right)^2 \to \left(\sum |z_i|^2\right)^2 + \frac{1}{2} P_1^2 \sum |z_i|^2 + \frac{1}{16} P_1^4,$$

$$\mathrm{Re}(z_1 z_2 z_3 z_4) \to \mathrm{Re}(z_1 z_2 z_3 z_4) - \frac{P_1^2}{16}(z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4) + \frac{P_1^4}{256} =$$

$$= \mathrm{Re}(z_1 z_2 z_3 z_4) - \frac{P_1^2 Q_1^2}{32} + \frac{P_1^2}{128} \sum Q_i^2 + \frac{P_1^4}{256}.$$

Expressing also (as always, z_i are old, defined by BBS (11.89))

$$\sum |z_i|^2 = \frac{1}{4} \sum Q_i^2,$$

and substituting results to BBS (11.88), one gets

$$\Delta \to \Delta - \frac{P_1^2 Q_1^2}{4}.$$

`Problem` **11.8**

There're several points about complex structure moduli space we are to recall. Complex structure moduli space is Kähler space with complex coordinates t^α and Kähler potential BBS (11.110) and metric tensor with components

$$G_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}. \tag{11.121}$$

Due to possible change of complex structure, Ω is not a holomorphic 3-form, but rather a sum of holomorphic 3-form with complex structure $(2,1)$-forms. This idea may be expressed as

$$\partial_\alpha \Omega = K_\alpha \Omega + \chi_\alpha. \tag{11.122}$$

According to BBS (9.102) complex structure metric is given by

$$G_{\alpha\bar{\beta}} = -\frac{\int \chi_\alpha \wedge \bar{\chi}_{\bar{\beta}}}{\int \Omega \wedge \bar{\Omega}}.$$

Consistency with (11.121), (11.122) and BBS (11.110) requires, as pointed on the top of p. 395 BBS, that

$$K_\alpha = -\partial_\alpha \mathcal{K}.$$

From (11.122) and BBS (11.110) it follows that this condition is equivalent to

$$\int \chi_\alpha \wedge \bar{\Omega} = 0. \tag{11.123}$$

This constraint also may be accompanied with

$$\int \chi_\alpha \wedge \Omega = 0, \tag{11.124}$$

which is a consequence of the fact that integrated form has not any $(3,3)$-terms whatsoever. Finally, note that Kähler covariant derivative is defined to satisfy

$$\mathcal{D}_\alpha \Omega = \chi_\alpha.$$

Rewrite BBS (11.120) in a form (we will integrate this over compact CY_3 manifold, and therefore don't pay attention that equality is defined only up to exact-form terms)

$$\frac{d}{d\tau}\left[e^{-U}(e^{-i\alpha+\mathcal{K}/2}\Omega - e^{i\alpha+\mathcal{K}/2}\bar{\Omega})\right] = -i\Gamma.$$

Integration over CY_3 of the wedge product of both sides with $e^{-i\alpha+\mathcal{K}/2}\mathcal{D}_\alpha\Omega = e^{-i\alpha+\mathcal{K}/2}\chi_\alpha$ gives equality

$$\mathcal{L} = \mathcal{R},$$

where

$$\mathcal{L} = e^{-i\alpha+\mathcal{K}/2}\int \chi_\alpha \wedge \frac{d}{d\tau}\left[e^{-U}(e^{-i\alpha+\mathcal{K}/2}\Omega - e^{i\alpha+\mathcal{K}/2}\bar{\Omega})\right],$$

$$\mathcal{R} = -ie^{-i\alpha+\mathcal{K}/2}\int \chi_\alpha \wedge \Gamma. \tag{11.125}$$

Due to (11.123) and (11.124) most of terms in the expression for $\mathcal{L}$ vanish after integration. Non-vanishing term contains

$$\int \chi_\alpha \wedge \frac{d}{d\tau}\bar{\Omega} = \frac{d\bar{t}^{\bar{\beta}}}{d\tau}\int \chi_\alpha \wedge \partial_{\bar{\beta}}\bar{\Omega} = \frac{d\bar{t}^{\bar{\beta}}}{d\tau}\int \chi_\alpha \wedge (\bar{K}_{\bar{\beta}}\bar{\Omega} + \bar{\chi}_{\bar{\beta}}) =$$

$$= \frac{d\bar{t}^{\bar{\beta}}}{d\tau}\int \chi_\alpha \wedge \bar{\chi}_{\bar{\beta}} = -\frac{d\bar{t}^{\bar{\beta}}}{d\tau}G_{\alpha\bar{\beta}}\int \Omega \wedge \bar{\Omega}.$$

Therefore one finds

$$\mathcal{L} = -e^{-i\alpha+\mathcal{K}/2}\int \chi_\alpha \wedge e^{i\alpha+\mathcal{K}/2-U}\frac{d}{d\tau}\bar{\Omega} = e^{\mathcal{K}-U}\frac{d\bar{t}^{\bar{\beta}}}{d\tau}G_{\alpha\bar{\beta}}\int \Omega \wedge \bar{\Omega}.$$

Here we may also replace

$$\int \Omega \wedge \bar{\Omega} = -ie^{-\mathcal{K}},$$

which will give

$$\mathcal{L} = -ie^{-U}\frac{d\bar{t}^{\bar{\beta}}}{d\tau}G_{\alpha\bar{\beta}}. \tag{11.126}$$

Observe that

$$\partial_{\bar{\beta}}|Z| = \frac{\partial_{\bar{\beta}}|Z|^2}{2|Z|}. \tag{11.127}$$

From the other side according to BBS (11.114)

$$\partial_{\bar{\beta}}|Z|^2 = \partial_{\bar{\beta}}\left(e^{\mathcal{K}}\int \Gamma \wedge \Omega \int \bar{\Gamma} \wedge \bar{\Omega}\right) =$$

$$= e^{\mathcal{K}}\int \Gamma \wedge \Omega\left(-\bar{K}_{\bar{\beta}}\int \bar{\Gamma} \wedge \bar{\Omega} + \int \bar{\Gamma} \wedge \partial_{\bar{\beta}}\bar{\Omega}\right) =$$

$$= e^{\mathcal{K}} \int \Gamma \wedge \Omega \left(-\bar{K}_{\bar{\beta}} \int \bar{\Gamma} \wedge \bar{\Omega} + \int \bar{\Gamma} \wedge (\bar{K}_{\bar{\beta}} \bar{\Omega} + \bar{\chi}_{\bar{\beta}}) \right) =$$

$$= e^{\mathcal{K}} \int \Gamma \wedge \Omega \int \bar{\Gamma} \wedge \bar{\chi}_{\bar{\beta}}.$$

Therefore due to (11.127) one gets

$$\partial_{\bar{\beta}} |Z| = \frac{1}{2|Z|} e^{\mathcal{K}} \int \Gamma \wedge \Omega \int \bar{\Gamma} \wedge \bar{\chi}_{\bar{\beta}}.$$

Therefore performing complex conjugation one results in

$$\int \Gamma \wedge \chi_\beta = 2|Z| e^{-\mathcal{K}} \partial_\beta |Z| \frac{1}{\int \bar{\Gamma} \wedge \bar{\Omega}}.$$

From BBS (11.114) it follows that

$$\int \bar{\Gamma} \wedge \bar{\Omega} = \frac{e^{-i\alpha} |Z|}{e^{\mathcal{K}/2}}.$$

Therefore

$$\int \Gamma \wedge \chi_\beta = 2 e^{i\alpha - \mathcal{K}/2} \partial_\beta |Z|.$$

Using this in (11.125) one gets

$$\mathcal{R} = 2i \partial_\alpha |Z|.$$

Together with (11.126) this implies

$$e^{-U} \frac{d\bar{t}^{\bar{\beta}}}{d\tau} G_{\alpha\bar{\beta}} = -2\partial_\alpha |Z|.$$

Contraction with inverse metric and complex conjugation immediately bring us to

$$\frac{dt^\alpha}{d\tau} = -2 e^U G^{\alpha\bar{\beta}} \partial_{\bar{\beta}} |Z|,$$

which is BBS (11.119).

`Problem` **11.9**

The key formula here is BBS (9.116), which in this chapter was used for derivation of BBS (11.115):

$$\int_M \Omega \wedge \bar{\Omega} = -\sum_I \left(\int_{A^I} \Omega \int_{B_I} \bar{\Omega} - \int_{A^I} \bar{\Omega} \int_{B_I} \Omega \right).$$

Insert definitions BBS (9.107), (9.109) of X and F coordinates (we don't use rescaled BBS (11.112)) into this formula:

$$\int_M \Omega \wedge \bar{\Omega} = -(X^I \bar{F}_I - \bar{X}_I F^I) = -2i \mathrm{Im}(X^I \bar{F}_I).$$

Therefore from

$$\mathcal{K} = -\log\left(i\int_M \Omega \wedge \bar{\Omega}\right)$$

one gets

$$\mathcal{K} = -\log\left(2\mathrm{Im}(X^I \bar{F}_I)\right).$$

According to BBS (9.113), (9.115), rescaling $X \to \lambda X$ with corresponding $F_I \to \tilde{F}_I$ to be found means

$$\lambda^2 F(x) = \frac{1}{2}\lambda X^I \tilde{F}_I,$$

therefore it must be $\tilde{F}_I = \lambda F_I$. Therefore if we perform a change of variables with $I > 0$ (these indices are denoted as α and correspond to $(2,1)$-forms, while $I = 0$ corresponds to $(3,0)$-form): $X^\alpha = t^\alpha X^0$, then we are to replace F_I by $\tilde{F}_\alpha = \frac{1}{X^0}F_\alpha$. Therefore we will result in

$$\mathcal{K} = -\log\left(2\mathrm{Im}(X^0 \bar{F}_0 + t^\alpha \bar{\tilde{F}}_\alpha)\right).$$

`Problem` **11.10**

Rotating supersymmetric $D = 5$ three-charge black hole is given by metric BBS (11.74), where 3-sphere metric BBS (11.73) is written in Hopf coordinates BBS (11.70)-(11.71). Let's compare it to BPS black hole BBS (11.142). This gives (there's a typo in first round bracket of BBS (11.74), should be two minus signs)

$$f = \lambda^{\frac{1}{3}} = \prod_{A=1}^{3}\left[1 + \left(\frac{r_A}{r}\right)^2\right]^{\frac{1}{3}},$$

$$\omega = \frac{a}{r^2}(-\cos^2\theta d\psi - \sin^2\theta d\phi),$$

$$ds_X^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2).$$

The most general BPS black hole, coupled to vector multiplets (each multiplet is labeled by index A) with $U(1)$ field strengths' F_A and scalars Y^A, must satisfy conditions BBS (11.144)-(11.147). In our case we have three charges. If we define Θ^A to be zero (which automatically satisfies BBS (11.145)) and Y^A in such a way that

$$fY^A = 1 + \left(\frac{r_A}{r}\right)^2,$$

then BBS (11.147), which in our case reduces to $\nabla^2(fY^A) = 0$ with four-dimensional spatial Laplacian, is satisfied because n-dimensional spherically symmetric Laplace equation in space with n coordinates is solved by r^k with $k = 2 - n$. Because of now we know both f and Y, we can determine field strength F^A by BBS (11.144).

Then one can easily observe that Hodge duality in flat four dimensions with metric ds_X^2, given by

$$\star(dx^{i_1} \wedge dx^{i_2}) = \frac{\varepsilon^{i_1 i_2 i_3 i_4}}{r^3\sin(2\theta)} g_{i_3 j_3} g_{i_4 j_4} dx^{j_3} \wedge dx^{j_4},$$

where we have used the fact that $\theta \in [0, \frac{\pi}{2}]$, results in

$$\star_4 d\omega = -d\omega,$$

which also employs the fact that positive-orientation order of Hopf coordinates is (r, θ, ϕ, ψ). This result agrees with BBS (11.146) condition.

`Problem` **11.13**

In this problem we are to count the number of states in bosonic left-moving sector of heterotic string with account to degeneration of total vacuum. The later is equal to 16 - the number of Majorana-Weyl spinor components of vacuum state of $\mathcal{N} = 1$ $D = 10$ supersymmetric right-moving sector. Degeneration of heterotic string state with left-moving excitation number N_L is given by coefficient in front of q^{N+1}, corresponding to given N_L by $N = N_L - 1$, in power expansion

$$\sum d_N q^{N+1}$$

of

$$16\text{tr}q^{\hat{N}_L} = 16 \prod_{n=1}^{\infty} \prod_{i=1}^{24} \text{tr}q^{\alpha_{-n}^i \alpha_n^i} = 16 \left[\prod_{n=1}^{\infty} (1 - q^n) \right]^{-24},$$

where trace is performed over all string excitation states in all possible transversal directions and we use infinite geometric progression summation formula for each transverse direction on equal footing. As a result, coefficient in front of q^{N_L} is composed out of all possible string states which are eigenstates of $\hat{N}_L$ with equal eigenvalues N_L. Introducing definition

$$Z = \sum d_N q^N,$$

we will get

$$Z = \frac{16}{q} \left[\prod_{n=1}^{\infty} (1 - q^n) \right]^{-24},$$

which with account to definitions BBS (11.163), (11.164) gives BBS (11.162), as required.

`Problem` **11.15**

Consider Einstein-Hilbert Lagrangian

$$\mathcal{L} = \frac{1}{16\pi} R\sqrt{-g} = \frac{1}{16\pi} \sqrt{-g} g^{\mu\nu} g^{\rho\lambda} R_{\rho\mu\lambda\nu}. \tag{11.128}$$

According to Wald's formula entropy is given by integral over horizon:

$$S = 2\pi \int d\Omega E_{\rho\mu} E_{\lambda\nu} \frac{\delta \mathcal{L}}{\delta R_{\rho\mu\lambda\nu}},$$

where $E_{\mu\nu} = \sqrt{g_2}\varepsilon_{\mu\nu}$ is antisymmetric Levi-Chivita tensor on $D = 2$ S^2 spatial part of the horizon surface, with g_2 being determinant of induced horizon metric. As soon as

$$\varepsilon_{\rho\mu}\varepsilon_{\lambda\nu} g^{\mu\nu} g^{\rho\lambda} = \varepsilon^{\lambda\nu} \frac{1}{g_2} \varepsilon_{\nu\lambda} = \frac{2}{g_2},$$

and when we restrict our consideration only towards horizon geometry we are to replace $\sqrt{-g} \to \sqrt{g_2}$ in (11.128), then we have

$$S = \frac{1}{4} \int \sqrt{g_2} d\Omega = \frac{A}{4},$$

which is Bekenstein-Hawking formula.

`Problem` **11.16**

We consider non-extremal three-charge black hole in $D = 5$ space-time, realized as bound state of Q_1 $D1$-branes wound around y^1 compact direction, Q_2 $D5$-branes wound around $y^1 \cdots y^5$ compact directions, n left-moving values of KK excitations and all corresponding antibranes with negative charges and KK momentum $\bar{n}$ of right-movers. Then if we consider some $F1$-string, starting on one of $D1$- or $\bar{D}1$-brane and ending on one of $D5$- or $\bar{D}5$-brane, then according to level-matching condition BBS (6.13) one will have

$$nW = N_L - N_R.$$

Namely if we consider left-movers with KK excitation number n, then $N_R = 0$, and in the case of right-movers we have KK excitation number $\bar{n}$ and condition $N_L = 0$ for short supermultiplet. Anyway, for modulus of N_L or N_R we will have the value nW. We have 4 different kinds of $F1$-string with W equal to one of the following values:

$$Q_1 Q_5, \quad Q_1 \bar{Q}_5, \quad \bar{Q}_1 Q_5, \quad \bar{Q}_1 \bar{Q}_5.$$

Considered here $F1$-strings leave in compact directions, and therefore each such string has 4 transverse physical directions, which according to supersymmetry requires 4 fermions. Each $F1$-string may be considered as independent physical subsystem, and total entropy may be considered as sum of entropies of each of the subsystem. For modulus of $N = N_L + N_R$ we have the value nW, which according to string with world-sheet supersymmetry R-sector mass formula BBS (4.109) (which contains number of excitation operators we are interested in here) gives for each $F1$-string formula BBS (11.95). After this formula is retrieved, all microscopic calculations of entropy are described on pp. 584-585 BBS. Therefore in the leading order entropy is the sum of terms for different cases of $F1$-strings:

$$S = 2\pi \sum \sqrt{Q_i Q_j Q_k} \tag{11.129}$$

where $Q_i,\ Q_j,\ Q_k$ takes on of the following 8 values

$$n \times (Q_1 Q_5, \quad Q_1 \bar{Q}_5, \quad \bar{Q}_1 Q_5, \quad \bar{Q}_1 \bar{Q}_5),$$

$$\bar{n} \times (Q_1 Q_5, \quad Q_1 \bar{Q}_5, \quad \bar{Q}_1 Q_5, \quad \bar{Q}_1 \bar{Q}_5).$$

Obviously this may be rewritten as

$$S = 2\pi \prod_{i=1}^{3} (\sqrt{Q_i} + \sqrt{\bar{Q}_i}),$$

which is BBS (11.69).

Problem 11.17

Consider $D = 4$ black hole in type-IIA superstring theory with six compact directions and four types of charges. Construction of black hole may be understood in terms of the following set of branes (notation: number, type, directions wrapped by this brane or KK excitation):

$$Q_1 \;\; D2 \;\; (y^1,\, y^6), \quad Q_2 \;\; D6 \;\; (y^1 \cdots y^6), \quad Q_3 \;\; NS5 \;\; (y^1 \cdots y^5), \quad Q_4 \;\; P \;\; y^1.$$

Performing T-duality in y^6 direction we will get representation of black hole as

$$Q_1 \;\; D1 \;\; y^1, \quad Q_2 \;\; D5 \;\; (y^1 \cdots y^5), \quad Q_3 \;\; NS5 \;\; (y^1 \cdots y^5), \quad Q_4 \;\; P \;\; y^1.$$

Introduce fundamental $F1$-string, localized in non-compact directions, connecting $D1$-brane and $D5$-brane and wrapped around closed y^1-loop. To start and end at the same point of compact submanifold T^6, it should go around y^1 compact direction Q_1Q_2 times (assuming that there're no common factors for Q_1 and Q_2). From the other side, while fundamental $F1$-string is able to end on $D5$-brane, S-dual picture requires $D1$-string to be able to end on $NS5$-brane. Therefore we actually should go around closed y^1-loop $Q_1Q_2Q_3$ times for $D1$-brane to start and end at the same point of $NS5$-brane (again, assuming no common factors for Q_1Q_2 and Q_3). As a result, we will get fundamental $F1$-string with winding number $W = Q_1Q_2Q_3$ and KK excitation number Q_4. For short supersymmetry either N_L or N_R for $F1$-string should vanish, and therefore level-matching condition

$$N_L - N_R = nW,$$

where n is KK excitation number, will give the value $|nW|$ for either N_L or N_R. As soon as considered $F1$-string has five transverse bosonic excitation degrees of freedom, it should have five fermionic degrees of freedom, and then R-sector number operator is expressed as

$$N = \sum_{m=1}^{\infty} \sum_{i=1}^{5} (N_m^i + mn_m^i),$$

where N_m^i is bosonic number operator for string excitation level m in transverse direction i, and similar for fermionic operator mn_m^i. The value of N is equal to either N_L or N_R. Therefore we have formula for level-matching condition analogous to BBS (11.95):

$$Q_1Q_2Q_3Q_4 = \sum_{m=1}^{\infty} \sum_{i=1}^{5} (N_m^i + mn_m^i).$$

The degeneracy d of state of constructed $F1$-string is given by number of ways to construct state with $|N| = |Q_1Q_2Q_3Q_4|$ times degeneration $N_0 = 16$ (for type-II superstring) of ground state. The value of d is given by coefficient of w^{WQ_4} in expansion of generating function

$$G(w) = N_0 \prod_{m=1}^{\infty} \left(\frac{1 + w^m}{1 - w^m} \right)^5.$$

Then calculation similar to that provided on p. 585 BBS gives formula (leading term in large WQ_4 limit)

$$S = 2\pi \sqrt{Q_1Q_2Q_3Q_4},$$

which is BBS (11.84).

Now let's study rotating three-charge black hole in five dimensions. Again, we perform counting of microstates of $F1$-string, which starts on $D1$-brane and ends on $D5$-brane. The black hole is described on p. 574 BBS, while D-brane and KK momentum set-up is the same as described on p. 583 BBS for three-charge black hole in five dimensions with the difference that now level-matching condition is not $L_0 - \bar{L}_0 = 0$, but $L_0 - \bar{L}_0 = J^2$. Indeed, now we are to describe rotating black hole, and therefore we must ascribe non-zero momentum $P = 2\pi(L_0 - \bar{L}_0)$ to $F1$-string, which is Noether current corresponding to world-sheet translations in spatial coordinate σ (this Noether current is classically used to define string physical states by zero action on them, but here our $F1$-string being non-critical - with central charge $c = 6$ not equal to 15 - breaks conformal symmetry anyway). For $F1$-string wound around closed y^1 coordinate this effectively looks like angular momentum J^2 of corresponding black hole. Therefore instead of BBS (11.95) we will have level-matching condition formula

$$|nW| = \sum_{i=1}^{4} \sum_{m=1}^{\infty} (N_m^i + m n_m^i) + J^2.$$

Thus to get entropy of rotating black hole we may simply use formula BBS (11.103) and replace there $|nW| \to |nW| - J^2$. This will give desired BBS (11.76).

12 Gauge theory/string theory dualities

`Problem` **12.1**
To solve this problem one must recall that ansatz BBS (12.4) for space-time configuration of $M2$-brane world-volume is of the same form as ansatz used in Chapter 10 for warp compactification of M-theory on CY_4 (formula BBS (10.5)). Indeed they differ by substitution $\Delta = H^{2/3}$. Supersymmetry constraints BBS (12.3) reduce to BBS (10.22), (10.24), (10.25), (10.27). Namely constraints related to 4-form flux and warp factor are then

$$\mathbf{F}\eta = 0, \quad \mathbf{F}_{\mathrm{m}}\xi = 0, \quad \mathrm{f_m} = \partial_{\mathrm{m}}\mathrm{H}^{-1} \tag{12.130}$$

where as in Chapter 10 η is spinor in $D = 8$ space, $\xi = H^{\frac{1}{6}}\eta$ and

$$\mathbf{F} = \frac{1}{4!}\mathrm{F_{mnpq}}\gamma^{\mathrm{mnpq}}, \quad \mathbf{F}_{\mathrm{m}} = \frac{1}{3!}\mathrm{F_{mnpq}}\gamma^{\mathrm{npq}}, \quad \mathrm{F}_{\mu\nu\rho\mathrm{m}} = \varepsilon_{\mu\nu\rho}\mathrm{f_m}.$$

Proposed 4-form flux BBS (12.5) leads to automatical satisfaction of constraints (12.130) due to $\mathbf{F}, \mathbf{F_m} = 0$, and $f_m = \partial_m H^{-1}$.

`Problem` **12.2**
To find temperature of non-extremal black Dp-brane, we keep only (t, r)-part of the metric BBS (12.31) and make change of variables: transition to Eucildean time

$$\tau = it$$

and expansion of radial coordinate in the vicinity of horizon $r = r_+$

$$r = r_+(1 + \rho^2).$$

As a result we get

$$ds^2 = \Delta_+ \Delta_-^{-1/2} d\tau^2 + 4r_+^2 \rho^2 \Delta_+^{-1} \Delta_-^{\gamma} d\rho^2.$$

Performing expansion of $\Delta_\pm$ products to get the form of interval similar to that deduced in the solution of Problem 11.5, we will result in

$$ds^2 = \frac{4r_+^2}{7-p} \left(1 - \left(\frac{r_-}{r_+}\right)^{7-p}\right)^{\gamma} \left(d\rho^2 + \left(\frac{7-p}{2r_+}\right)^2 \left(1 - \left(\frac{r_-}{r_+}\right)^{7-p}\right)^{-\frac{1}{2}-\gamma} \rho^2 d\tau^2\right).$$

Therefore one easily concludes that inverse temperature is given by

$$\beta = 2\pi \left(\frac{2r_+}{7-p} \left(1 - \left(\frac{r_-}{r_+}\right)^{7-p}\right)^{\frac{1}{4}+\frac{\gamma}{2}}\right),$$

which according to BBS (12.32) takes the form

$$\beta = 2\pi \left(\frac{2r_+}{7-p} \left(1 - \left(\frac{r_-}{r_+}\right)^{7-p}\right)^{-\frac{5-p}{2(7-p)}}\right). \tag{12.131}$$

This formula was deduced only for non-extremal black branes and is not applicable for extremal black branes. One can easily see it post factum, because it doesn't give zero temperature for all values of $p < 7$ when $r_- = r_+$. The reason is that $r = r_+(1+\rho^2)$ near-horizon approximation of radial coordinate, which was used to deduce non-extremal case formula (12.131), gives wrong form of metric in the extremal case (not the form of planar metric in angular coordinates) and therefore can't be used to make any conclusions about periodicity of Euclidean time and therefore about temperature.

The part of metric BBS (12.31) relevant for calculation of area of horizon is

$$ds^2 = \Delta_-^{1/2} dx^i dx^i + r^2 \Delta_-^{\gamma+1} d\Omega_{8-p}^2.$$

Horizon has topology $R^p \times S^{8-p}$, but we are to calculate entropy per unit volume of R^p, and therefore we must calculate only spherical part of area:

$$A = r_+^2 \Delta_-^{\gamma+1}(r_+) A_{8-p} = r_+^2 \left(1 - \left(\frac{r_-}{r_+}\right)^{7-p}\right)^{\gamma+1} \frac{2\pi^{\frac{9-p}{2}}}{\Gamma\left(\frac{9-p}{2}\right)}.$$

Entropy is given by formula

$$S = \frac{A}{4G_{10}},$$

where for even p gamma-function is to be calculated with the help of $\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$ and for odd p with the help of $\Gamma(n) = (n-1)!$

`Problem` **12.3**

Radius of $D = 4$ Schwarzshild black hole is given by $r_1 = 2G_4M_4$, where M_4 is "four-dimensional" mass. Total mass M of five-dimensional black string also takes into account mass distribution with some density ρ along the R-line of string. The length of compact line is $2\pi R$, therefore $M \sim r_1 R$. Area of event horizon is given by $A = (4\pi r_1^2)(2\pi R)$, and therefore entropy of black string is proportional to $S_1 \sim r_1^2 R$. At the same time according to BBS (11.11) mass of five-dimensional black hole is proportional to square of its radius r_0. Horizon is 3-dimensional surface with area of the order r_0^3. Therefore $S_0 \sim r_0^3$. Several five-dimensional black holes, which are supposed to be product of decay of one black string, have total mass and entropy (additive) of the same order as one of them. Equality of orders of masses of black holes and black string, required by energy conservation, gives $r_1 R \sim r_0^2$, and therefore $\frac{S_1}{S_0} = k\frac{r_0}{R}$. Observe that in this process 1-brane (black string) has decayed into several 0-branes (black holes).

`Problem` **12.4**

In this problem we consider matrix representation of $su(M|N)$ superalgebra. To do it note that corresponding group $SU(M|N)$ is defined as group of transformations of multiplet (b, f), consisting of M bosons forming vector b and N fermions forming vector f, which leave the norm of multiplet invariant:

$$b\bar{b} + f\bar{f} = \text{inv.} \tag{12.132}$$

infinitesimally we can build transformation

$$\begin{pmatrix} \delta b \\ \delta f \end{pmatrix} = i\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} b \\ f \end{pmatrix}, \tag{12.133}$$

where 2×2-block matrix X on the r.h.s. is matrix of infinitesimal parameters of transformation. We obviously have requirement for B and C to be matrices with fermionic entries, because variation of fermion should be fermion, while variation of boson should be boson. For the condition of symmetry (12.132) to be satisfied it also must be take place the following set of conditions

$$A^\dagger = A, \quad D^\dagger = D, \quad C^\dagger = B. \tag{12.134}$$

Indeed, these conditions mean that (12.133) is unitary transformation and therefore unitary norm (12.132) is preserved. It's also assumed the condition of vanishing of supertrace, which is not meaningful for invariance of (12.132), but which is essential for speciality of the supergroup.

From (12.133) applied to two transformations with different sets of parameters (labeled by index $i = 1, 2$) one can calculate Lie brackets (which is the same procedure as commuting matrices)

$$(\delta_1\delta_2 - \delta_2\delta_1)b = i(Ab + Bf),$$
$$(\delta_1\delta_2 - \delta_2\delta_1)f = i(Cb + Df),$$

where

$$A = -i([A_1, A_2] + B_1C_2 - B_2C_1), \tag{12.135}$$
$$B = i(A_2B_1 - A_1B_2 + B_2D_1 - B_1D_2), \tag{12.136}$$
$$C = i(C_2A_1 - C_1A_2 + D_2C_1 - D_1C_2), \tag{12.137}$$

$$D = -i([D_1, D_2] + C_1 B_2 - C_2 B_1). \tag{12.138}$$

One can easily verify that matrices (12.135)-(12.138) satisfy conditions of unitarity (12.134). One can also easily calculate that

$$\mathrm{Str}X = \mathrm{tr}A - \mathrm{tr}D = i(\mathrm{tr}(B_2 C_1 + C_1 B_2) - \mathrm{tr}(B_1 C_2 + C_2 B_1)) = 0,$$

where we've used anticommutativity of *elements* of fermionic matrices B and C together with possibility of cyclic permutation inside trace (which therefore for fermionic matrices means antisymmetric cyclic permutation inside trace).

Finally observe that condition $\mathrm{Str}X = 0$, that is $\mathrm{tr}A = \mathrm{tr}D$, makes connection between two $U(1)$ transformations - invariant subalgebras of $\{A\}$ and $\{D\}$ algebras (each such subalgebra is one-parametric). Therefore this condition leaves us only with one independent $U(1)$ transformation. Each $u(1)$ algebra element in $n \times n$-matrix representation as element of $n \times n$ unitary matrix group ($n = N, M$) is a unit matrix times some complex number with unit modulus. Condition $\mathrm{Str}X = 0$ relates these numbers for two $u(1)$ elements. But if $M = N$, then this condition would require both $u(1)$ elements to be exactly the same. Therefore we will get one independent $u(1)$ element for total X-matrix. This is again unit matrix times some complex number with unit modulus. This matrix commutes with all other matrices of X, hence it decouples from $SU(N|N)$.

`Problem` **12.5**

Spinor in $D = 5$ space-time has 8 components, therefore SYM theory with 16 supercharges in $D = 5$ possesses $\mathcal{N} = 2$ supersymmetry. We have 4 rising supersymmetry generators for massless SYM multiplet. Spin field content is

$$|-1\rangle^1 \quad |-1/2\rangle^4 \quad |0\rangle^6 \quad |1/2\rangle^4 \quad |1\rangle^1,$$

each field is in adjoint representation of $SU(N)$. R-symmetry group is $SU(4) \sim SO(6)$.

`Problem` **12.6**

As it's shown in the solution of Problem 12.7 the metric BBS (12.97) of AdS_{d+1} in Poincare coordinates may be rewritten in terms of $d+2$-dimensional Minkowski space-time coordinates on hypersurface BBS (12.98):

$$ds^2 = \sum_{i=1}^{d} dy_i^2 - \sum_{j=1,2} dt_j^2.$$

We can introduce spherical coordinates on spatial and time parts separately to rewrite the metric on that parts in the following form:

$$\sum_{i=1}^{d} dy_i^2 = dv^2 + v^2 d\Omega_p^2,$$

$$\sum_{j=1,2} dt_j^2 = d\tau^2 + \tau^2 d\theta^2.$$

Here dv and $d\tau$ are elements of radial distances; $d\Omega_p$ (where $p = d-1$) and $d\theta$ are elements of angular distances. Evidently BBS (12.98) may be recast as

$$v^2 - \tau^2 = -1,$$

from which we can easily express τ and $d\tau$ through v and dv to get

$$dv^2 - d\tau^2 - \tau^2 d\theta^2 = \frac{dv^2}{1+v^2} - (1+v^2)d\theta^2.$$

Now the origin of BBS (12.101) is clear.

`Problem` **12.7**

In metric BBS (12.97) (with $R = 1$) perform coordinate change

$$z = \frac{1}{t_1 + y_d} \qquad dz = -\frac{dt_1 + dy_d}{(t_1 + y_d)^2},$$

$$x^0 = \frac{t_2}{t_1 + y_d} \qquad dx^0 = \frac{dt_2}{t_1 + y_d} - \frac{t_2(dt_1 + dy_d)}{(t_1 + y_d)^2},$$

$$x^i = \frac{y_i}{t_1 + y_d} \qquad dx^i = \frac{dy_i}{t_1 + y_d} - \frac{y_i(dt_1 + dy_d)}{(t_1 + y_d)^2},$$

and take into account that

$$y_1^2 + \cdots + y_d^2 - t_1^2 - t_2^2 = -1,$$

$$y_1 dy_1 + \cdots y_d dy_d - t_1 dt_1 - t_2 dt_2 = 0.$$

As a result starting from BBS (12.97) one will evidently get

$$ds^2 = (t_1 + y_d)^2 \Big(\frac{-dt_2^2 + \sum_{i=1}^{d-1} dy_i^2}{(t_1 + y_d)^2} + \frac{(dt_1 + dy_d)^2}{(t_1 + y_d)^4} \left(1 - t_2^2 + \sum_{i=1}^{d-1} y_i^2 \right) -$$

$$- \frac{2(dt_1 + dy_d)}{(t_1 + y_d)^3} \sum_{i=1}^{d-1} y_i dy_i + \frac{2t_2 dt_2 (dt_1 + dy_d)}{(t_1 + y_d)^3} \Big) =$$

$$= -dt_1^2 - dt_2^2 + \sum_{i=1}^{d} dy_i^2,$$

which is metric in flat $2 + d$ space-time. Transition backwards to Poincare patch metric BBS (12.97) takes into account constraint BBS (12.98) and therefore describes embedding of AdS_{d+1} into flat $d + 2$-dimensional space-time.

`Problem` **12.8**

We can read Lagrangian of scalar field in $EAdS_5$ from the action BBS (12.115):

$$\mathcal{L} = \frac{1}{z^3} \left((\partial_y \phi)^2 + (\partial_z \phi)^2 \right) + \frac{1}{z^5} m^2 R^2 \phi^2.$$

Equation of motion in curved space-time is

$$(\Delta - m^2 R^2)\phi = 0,$$

with D'Alambertian

$$\Delta\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) = z^5\partial_\mu(z^{-3}\partial_\mu\phi) = z^2(\partial_z^2\phi + \partial_y^2\phi) - 3z\partial_z\phi,$$

where as always square of y-related terms means summation over all four y coordinates. Therefore equation of motion looks like

$$z^2(\partial_z^2\phi + \partial_y^2\phi) - 3z\partial_z\phi - m^2R^2\phi = 0.$$

Let's represent field configuration as

$$\phi(z, y) = e^{i\mathbf{p}\cdot\mathbf{y}}\tilde{\phi}(z),$$

which will evidently lead to equation

$$z^2\partial_z^2\tilde{\phi} - 3z\partial_z\tilde{\phi} - (m^2R^2 + z^2)\tilde{\phi} = 0.$$

This equation has general solutions in terms of Bessel functions with assymtotics $z^{\alpha_\pm}$ as $z \to 0$, where $\alpha_\pm$ are given by BBS (12.117).

`Problem` **12.9**

We start with $AdS_5 \times S^5$ space-time with metric

$$g_{\mu\nu} = \text{diag}\{-\left(\frac{r}{R}\right)^2, \left(\frac{r}{R}\right)^2, \left(\frac{r}{R}\right)^2, \left(\frac{r}{R}\right)^2, \left(\frac{R}{r}\right)^2, R^2 g_{ij}\}, \tag{12.139}$$

where g_{ij} denotes diagonal metric components for unit sphere, both AdS_5 and S^5 have the same radius R. We know that if AdS_5 is sourced by 'heavy' stack of N $D3$-branes (on its boundary), which is the case here, then $R^4 = 4\pi g_s N\alpha'^2$ (see BBS (12.28), (12.29)). Introduce the value

$$f = \frac{R^4}{r^4}.$$

If we want to construct DBI action for probe $D3$-brane in $AdS_5 \times S^5$ background we first of all shall perform pullback of metric (12.139) on $D3$-brane world-volume coordinates. As far as geometric background is fixed, we can naturally choose first four space-time coordinates to parametrize $D3$-brane world-volume. We shall also take into account motion of $D3$ brane along r and S^5 coordinates. Therefore pullback metric on $D3$-brane world-volume is given by

$$g_{\alpha\beta} = f^{-1/2}\eta_{\alpha\beta} + f^{1/2}\partial_\alpha r\partial_\beta r + r^2 f^{1/2} g_{ij}\partial_\alpha\theta^i\partial_\beta\theta^j,$$

where we have used the fact that $r^2 f^{1/2} = R^2$ and denoted unit 5-sphere coordinates by θ^i. Bosonic part of free DBI action is given by formula BBS (6.106) with $\mathcal{F}_{\alpha\beta} = F_{\alpha\beta}$ and tension

$$T_{D3} = \frac{1}{(2\pi)^3\alpha'^2 g_s}$$

following from BBS (6.115). Then we result in action

$$S_1 = -\frac{1}{(2\pi)^3\alpha'^2 g_s}\int d^4x f^{-1}\sqrt{-\det\left(\eta_{\alpha\beta} + f\partial_\alpha r\partial_\beta r + r^2 f g_{ij}\partial_\alpha\theta^i\partial_\beta\theta^j + 2\pi\alpha'\sqrt{f}F_{\alpha\beta}\right)}.$$

Here x denote first four coordinates of AdS_5 - boundary coordinates. Determinant is taken for 4×4 matrix with indices $\alpha,\ \beta$. There's also a second term S_2 of total DBI action $S=S_1+S_2$ which describes coupling of $D3$-brane to field A_4:

$$S_2=\mu_3\int A_4.$$

As soon as $D3$-brane is BPS object then we have related charge and tension for it:

$$\mu_3=T_{D3}=\frac{1}{(2\pi)^3\alpha'^2 g_s}.$$

Due to symmetry considerations (this actually follows from exact solution BBS (12.25) of type-IIB supergravity equations) we choose $A_{\mu\nu\lambda\rho}=\sqrt{|g_4|}\varepsilon_{\mu\nu\lambda\rho}=f^{-1}\varepsilon_{\mu\nu\lambda\rho}$ and therefore we result in

$$S_2=\frac{1}{(2\pi)^3\alpha'^2 g_s}\int d^4x f^{-1}.$$

Our result for bosonic DBI action is then

$$S=-\frac{1}{(2\pi)^3\alpha'^2 g_s}\int d^4x f^{-1}\left[\sqrt{-\det\left(\eta_{\alpha\beta}+f\partial_\alpha r\partial_\beta r+r^2 f g_{ij}\partial_\alpha\theta^i\partial_\beta\theta^j+2\pi\alpha'\sqrt{f}F_{\alpha\beta}\right)}-1\right].$$

One can also eliminate explicit dependence of action on α' by simple change of variables $r=u\alpha'$, because $f\sim\alpha'^{-2}$. Studying AdS/CFT correspondence we are interested in low-energy limit, which may be equivalently achieved by sending the distanse between string excitation levels to infinity: $\alpha'\to 0$. At this limit bulk theory decouples from boundary theory because square root of Newton constant $\kappa\sim g_s\alpha'^2\to 0$. Energy E_b of any bulk excitation located near coordinate r gives the value $E=g_{tt}(r)E_b=\frac{r}{\alpha'}E_b$ when is measured by observer at infinity (due to redshift). If we hold energy E at infinity and energy E_b fixed in string units this will lead to requirement of fixed $u=r/\alpha'$. This is natural radial bulk coordinate for description of AdS_5/CFT_4 correspondence.

`Problem` **12.10**

Let's calculate the volume of cone base $T^{1,1}$. Due to BBS (10.120) one has a length element of $T^{1,1}$ defined by formula

$$d\Sigma^2=\frac{4}{9}d\beta^2+\frac{4}{9}\cos\theta_1 d\beta d\phi_1+\frac{4}{9}\cos\theta_2 d\beta d\phi_2+$$
$$+\frac{1}{9}\cos^2\theta_1 d\phi_1^2+\frac{1}{9}\cos^2\theta_2 d\phi_2^2+\frac{2}{9}\cos\theta_1\cos\theta_2 d\phi_1 d\phi_2+$$
$$+\frac{1}{6}d\theta_1^2+\frac{1}{6}d\theta_2^2+\frac{1}{6}\sin^2\theta_1 d\phi_1^2+\frac{1}{6}\sin^2\theta_2 d\phi_2^2.$$

Well, one can write down 6×6 matrix for $(r,\beta,\theta_1,\phi_1,\theta_2,\phi_2)$ conifold now; of course, I will not do it here. I just note that it's convenient to represent

$$\frac{1}{9}\cos^2\theta_i+\frac{1}{6}\sin^2\theta_i=\frac{1}{9}\left(1+\frac{1}{2}\sin^2\theta_i\right).$$

Then calculating determinant one immediately reduces 6×6 matrix of metric to 5×5 matrix, which will have three terms of 4×4 matrices, each term with positive sign. Each 4×4 matrix will be proportional to one determinant of 3×3 matrix; first 4×4 matrix will reduce to such a matrix with positive-sign coefficient, the other two - with negative. An interesting feature is that *all* terms of complete expansion of total determinant give common multiplier $\frac{1}{9^4}$. Some trigonometry gives us then

$$\frac{1}{3^6\cdot 2^2}\sin^2\theta_1\sin^2\theta_2$$

for determinant of metric on $T^{1,1}$. The volume of $T^{1,1}$ is then given by

$$\mathrm{Vol}(T^{1,1}) = \int d\theta_1 d\theta_2 d\phi_1 d\phi_2 d\beta \frac{\sin\theta_1\sin\theta_2}{3^3\cdot 2} = \frac{16\pi^3}{27}.$$

As soon as volume of unit five-sphere (that is its surface area) is given by formula $\mathrm{Vol}(S^5)=\pi^3$, then we indeed result in formula

$$\mathrm{Vol}(T^{1,1}) = \frac{16}{27}\mathrm{Vol}(S^5).$$

`Problem` **12.15**

Variation of the action BBS (12.18) by p-form gauge field (which is represented only by its purely free field term) gives equation of motion

$$d\star F_{p+2} = 0, \tag{12.140}$$

where $\star F_{p+2}$ is $(8-p)$-form, Hodge dual to F_{p+2}. It's rather simple to check that $p+2$-form field strength given by BBS (12.22) satisfies (12.140). If p-brane world-volume coordinates are first $p+1$ coordinates of space-time, then positive-orientation order of coordinates on the whole space-time is

$$M_{10}: \quad (0,1,\dots,p,r,\phi_1,\dots,\phi_{8-p}),$$

where we've denoted unit sphere coordinates by ϕ_i. According to BBS (12.22) the only independent non-zero component of field strength is given by

$$F_{01\cdots pr} = (-1)^{p+1}(7-p)\frac{r_p^{7-p}}{r^{8-p}},$$

and therefore the only independent component of Hodge-dual field strength is equal to

$$(\star F)_{\phi_1\cdots\phi_{8-p}} = \sqrt{-g}\varepsilon_{01\cdots pr\phi_1\cdots\phi_{8-p}}F^{01\cdots pr} = (-1)^{(p+2)(8-p)}\sqrt{-g}F^{01\cdots pr}.$$

As soon as $(-1)^{(p+2)(8-p)} = (-1)^{p^2}$, $(-1)^{p^2+p}=1$ and

$$\sqrt{-g} = r^{8-p}H_p^{2-p/2}\sqrt{g_{8-p}},$$

where g_{8-p} is determinant of metric for unit $(8-p)$-sphere (function of ϕ_i coordinates), one has

$$(\star F)_{\phi_1\cdots\phi_{8-p}} = (p-7)r_p^{7-p}\sqrt{g_{8-p}},$$

and therefore

$$\star F = (p-7) r_p^{7-p} \sqrt{g_{8-p}} d\phi^1 \wedge \cdots \wedge d\phi^{8-p} = (p-7) r_p^{7-p} \omega_{8-p}.$$

Then obviously equation of motion (12.140) is satisfied.

Let's find equation of motion arising from variation of action BBS (12.18) by metric tensor. Due to variations

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu},$$

$$\delta(R\sqrt{-g}) = \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right)\sqrt{-g}\delta g^{\mu\nu},$$

$$\delta(\partial\Phi)^2 = \partial_\mu \Phi \partial_\nu \Phi \delta g^{\mu\nu},$$

$$\delta |F_{p+2}|^2 = \frac{1}{(p+1)!} F_{\mu\mu_1\cdots\mu_{p+1}} F_\nu{}^{\mu_1\cdots\mu_{p+1}} \delta g^{\mu\nu}.$$

variation of action by metric tensor is given by

$$\delta S^{(p)} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}(e^{-2\Phi}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 2g_{\mu\nu}(\partial\Phi)^2 + 4\partial_\mu\Phi\partial_\nu\Phi\right) +$$

$$+\frac{1}{4} g_{\mu\nu}|F_{p+2}|^2 - \frac{1}{2\cdot(p+1)!} F_{\mu\mu_1\cdots\mu_{p+1}} F_\nu{}^{\mu_1\cdots\mu_{p+1}})\delta g^{\mu\nu},$$

which gives stationary points by equations of motion

$$e^{-2\Phi}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 2g_{\mu\nu}(\partial\Phi)^2 + 4\partial_\mu\Phi\partial_\nu\Phi\right) + \frac{1}{4}g_{\mu\nu}|F_{p+2}|^2 - \frac{1}{2\cdot(p+1)!}F_{\mu\mu_1\cdots\mu_{p+1}}F_\nu{}^{\mu_1\cdots\mu_{p+1}} = 0.$$

Variation of the action by dilaton field gives

$$\delta S^{(p)} = -\frac{1}{\kappa^2}\int d^{10}x\sqrt{-g}e^{-2\Phi}(R + 4\Delta\Phi - 4(\partial\Phi)^2)\delta\Phi,$$

where as usual

$$\Delta\Phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\Phi).$$

Equations of motion are then

$$R + 4\Delta\Phi - 4(\partial\Phi)^2 = 0.$$

`Problem` **12.18**

Some contours of proof are already sketched in the solution of Exercise 12.3. Let's state details. According to BBS (12.63) (applied for $\mu^2 = b^2$) with account to definition BBS (12.62) $r^2 = b^2 + (v\tau)^2$ one has the following equation on bosonic propagator (in $0+1$ space-time):

$$(-\partial_\tau^2 + r^2)\Delta_{\mathcal{B}}(\tau, \tau'|r^2) = \delta(\tau - \tau'),$$

which can be rewritten is a form with redefined mass parameter:

$$(-\partial_\tau^2 + r^2 - v\gamma_1)\Delta_{\mathcal{B}}(\tau, \tau'|r^2 - v\gamma_1) = \delta(\tau - \tau'). \tag{12.141}$$

Here γ_1 is supposed to take some its eigenvalue.

From fermionic side we have mass matrix $m_{\mathcal{F}} = v\tau\gamma_1 + b\gamma_2$, and therefore

$$m_{\mathcal{F}}^2 = (v\tau)^2 + b^2,$$

$$[\partial_\tau, m_{\mathcal{F}}] = v\gamma_1.$$

Therefore due to BBS (12.66) one readily gets

$$(-\partial_\tau + m_{\mathcal{F}})\Delta_{\mathcal{F}}(\tau, \tau'|m_{\mathcal{F}}) = (-\partial_\tau^2 + r^2 - v\gamma_1)\Delta_{\mathcal{B}}(\tau, \tau'|r^2 - v\gamma_1).$$

Due to (12.141) this leads to

$$(-\partial_\tau + m_{\mathcal{F}})\Delta_{\mathcal{F}}(\tau, \tau'|m_{\mathcal{F}}) = \delta(\tau - \tau').$$

References

[1] Becker, K., Becker, M., Schwarz, J.H.: *String Theory And M-theory*, Cambridge University Press (2006).

[2] Green, M.B., Schwarz, J.H., Witten, E.: *Superstring Theory*, two volumes, Cambridge University Press (1986).

[3] Polchinski, J.: *String Theory*, two volumes, Cambridge University Press (2002).

[4] Kaku, M.: *Introduction to Superstrings and M-theory*, Springer Verlag (1999).

[5] Di Francesco, P., Mathieu, P., Senechal, D.: *Conformal Field Theory*, Springer Verlag (1997).

[6] Kiritsis, E.: *String Theory in a Nutshell*, Princeton University Press (2007).

[7] Hawking, S.W., Ellis, G.F.R.: *The large scale structure of space-time*, Cambridge University Press (1999).

[8] Weinberg, S.: *Cosmology*, Oxford University Press (1999).